The Most General Classes of Tellegen Media Reducible to Simple Reciprocal Media: a Geometrical Approach

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Abstract

Duality mappings allow to transform a nonreciprocal achiral bi-isotropic medium (Tellegen medium) into a conventional reciprocal material leaving the free space invariant. In particular, the solutions of electromagnetic problems involving a single Tellegen medium and a vacuum can be found by applying the inverse duality transformation to the solution of the duality transformed problem wherein the materials are conventional reciprocal media. Here, based on a geometrical interpretation of duality transformations in the Riemann sphere, we derive the most general classes of Tellegen media that can be reduced to simple isotropic media with the same duality transformation. Moreover, it is shown that Tellegen media can be identified with the points of the Riemann sphere, and duality transformations are classified into different categories according to their geometrical actions on this sphere. We apply the developed theory to periodic structures formed by Tellegen media, showing how the wave propagation in these complex structures can be easily studied using

duality transformations. Furthermore, to unveil the role of the nonreciprocal response, we investigate the wave propagation in Tellegen periodic structures wherein the pertinent materials are irreducible to conventional media.

1. Introduction

Duality transformations were originally defined as linear transformations of the electromagnetic field vectors [1-3]. These transformations have the ability to take a nonreciprocal achiral bi-isotropic medium (Tellegen or axion medium) [4] to a simple isotropic medium (SIM) leaving the vacuum invariant [5-8]. The duality transformed fields satisfy the Maxwell's equations in a transformed medium with constitutive parameters that are a linear combination of those of the original Tellegen medium. This property is rather useful when analytical solutions for the transformed problem are available [5]. Moreover, under more restrictive conditions bi-anisotropic media can also be reduced to simpler reciprocal media using a duality mapping [9].

Because of the possibility of reducing a Tellegen medium to a conventional dielectric, some authors claimed some time ago that Tellegen media are not uniquely recognizable [8]. The Post constraint [2] was the central argument in this controversy, and allegedly, according to these authors, it should be satisfied by all natural materials. However, other researchers experimentally demonstrated the realization of Tellegen artificial media [10] that violates the Post constraint. More recently, several studies [11-12] of the natural composite called chromium sesquioxide, Cr_2O_3, provided further support for the reality of Tellegen media. In addition, it was suggested that topological insulators can have in some circumstances an electromagnetic response consistent with that of axion media [13, 14].

Duality transformations are also useful in electromagnetic problems involving opaque boundaries [15]. Recently, a new electromagnetic boundary named perfect electromagnetic conductor (PEMC) [16-18] was introduced by Lindell and Sihvola generalizing other two boundaries: the perfect electric conductor (PEC) and the perfect magnetic conductor (PMC). The PEMC may also be seen as a special case of Minkowskian isotropic media (MIM) [19] which is the most general family of Tellegen media that is invariant under a Lorentz transformation. Using duality transformations, it was found that a waveguide filled by vacuum with PEMC walls and can be reduced to a similar structure but with PEC walls [15].

In this work, we give a geometrical interpretation for the duality transformations applied to Tellegen media. Furthermore, it is proven that isorefractive Tellegen media are isomorphic to the Riemann sphere [20]. We demonstrate that some particular cases of Tellegen media with special interest correspond to the meridians and poles of the unit sphere. Based on a geometrical interpretation, duality transformations are classified as generalized rotations, Lorentz boosts or Galilean boots. Moreover, duality transformations are identified with Möbius mappings [21] and are shown to be conformal transformations over the unit sphere. The developed geometrical concepts allow us to derive the most general classes of Tellegen media that are reducible to simple isotropic media by a suitable duality transformation. As an example of the application of this result, we analyze the wave propagation problem in periodic structures formed by Tellegen media. It is demonstrated that periodic structures formed by two Tellegen media can always be reduced to periodic structures formed by two simple isotropic materials. In addition, we also consider Tellegen periodic structures wherein the relevant media cannot be reduced to simpler media by a duality transformation. It is shown that the corresponding band diagrams are not topologically equivalent to those of conventional dielectrics, unveiling in this manner the signature of the nonreciprocal material response.

2. Tellegen Media in the Riemann Sphere

In this section, Tellegen media are geometrically represented in the Riemann sphere. In particular, some special cases of these media are identified as poles and meridians of the unit sphere.

Tellegen media are achiral nonreciprocal bi-isotropic media [4]. It is convenient to write the constitutive relations of a lossless Tellegen medium as

$$\eta_0\,\mathbf{D} - M\,\mathbf{B} = (|n|/c)\,\Gamma\,\mathbf{E}, \quad \eta_0\,\mathbf{H} + M\,\mathbf{E} = (c/|n|)\,\Gamma\,\mathbf{B}, \tag{1}$$

where η_0 is the vacuum wave impedance and $n = \sqrt{\varepsilon\,\mu - \kappa^2}$ is the refraction index. The admittance $M = \kappa/\mu$ is associated with the magnetoelectric coupling whereas the admittance $\Gamma = |n|/\mu$ gives the principal part of (1). On the other hand, the constitutive parameters and the wave impedance are parameterized as

$$\varepsilon = |n|\left(\Gamma^2 + M^2\right)/\Gamma, \quad \mu = |n|/\Gamma, \quad \kappa = |n|\,M/\Gamma, \quad \eta = \sqrt{1/\left(M^2 + \Gamma^2\right)}. \tag{2}$$

It is clear from (2) that any family of isorefractive Tellegen media, with a constant $|n|$, can be represented in the $\left(M, \Gamma\right)$ plane. Thus, using the stereographic projection $\hat{P}_r$ [20] it is possible to map each point in the complex plane $w = M + i\Gamma$ into a point $p\,(x, y, z)$ on the Riemann sphere [see Fig. 1a]. Therefore, any point on the Riemann sphere corresponds to a certain Tellegen medium which can be can be simply written in terms of two angles (χ, ϕ):

$$\varepsilon = |n|\left[\sec(\chi) + \cos(\phi)\tan(\chi)\right], \quad \mu = |n|\left[\sec(\chi) - \cos(\phi)\tan(\chi)\right], \quad \kappa = |n|\left[\sin(\phi)\tan(\chi)\right], \tag{3}$$

where the polar axis is aligned with the *y*-direction [see Fig. 1b].

Some particular cases of Tellegen media with special interest [19] correspond to specific limits of the admittances M and Γ. For example, the condition $M = 0$ yields the SIMs, and the condition $\Gamma = 0$ corresponds to the PEMCs. Thus, the SIMs and the PEMCs are represented by lines in the complex plane, and by meridians in the Riemann sphere as illustrated in Fig. 1c. The PEC and the PMC cases defined by $\eta = 0$ and $\eta \to \infty$, respectively, are the north and south poles of the unit sphere. In addition, the points on the hemisphere $y > 0$ correspond to Tellegen media with positive ε and μ parameters, whereas the region $y < 0$ is associated with media with negative values of ε and μ, i.e. with generalized nonreciprocal Veselago media with negative refractive index [22-23].

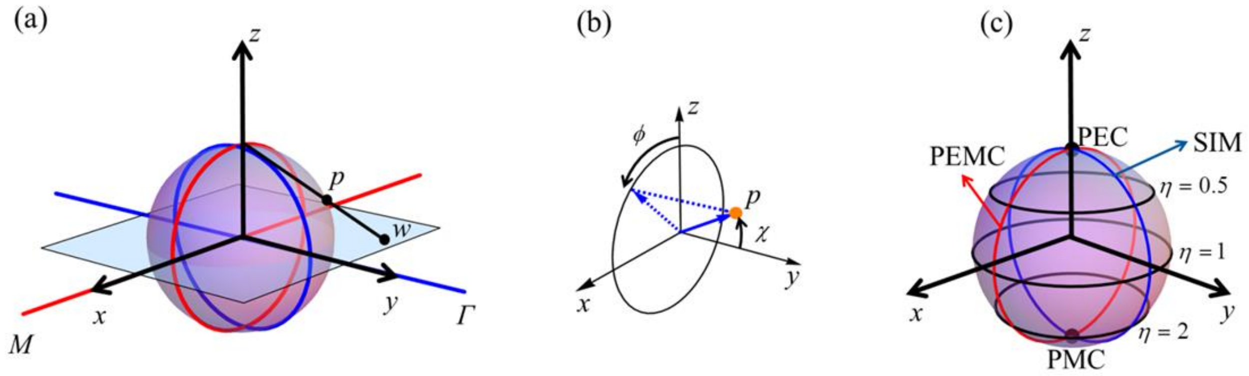

Fig. 1 (a) The stereographic projection. (b) The spherical coordinates of a point p on the surface of the Riemann sphere. (c) Geometrical representation of Tellegen media.

3. Duality Transformations in the Riemann Sphere

In this section, we investigate the action of duality transformations in the Riemann sphere. Depending on the specific geometrical actions, we classify the duality transformations into three distinct families. Each family is uniquely characterized by its fixed points, i.e, the Tellegen media that remain invariant under the transformation.

Duality transformations [1-3] are defined as linear mappings of the electromagnetic fields of the form:

$$\begin{pmatrix} \mathbf{E}_d \\ \eta_0 \mathbf{H}_d \end{pmatrix} = \bar{\mathbf{S}} \cdot \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix}, \text{ with } \bar{\mathbf{S}} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \tag{4}$$

where $\bar{\mathbf{S}}$ is a 2×2 real-valued matrix with constant elements (independent of the spatial coordinates) and s_{11}, s_{12}, s_{21} and s_{22} are real-valued parameters. It is well-known that the duality transformed fields $\mathbf{E}_d$ and $\mathbf{H}_d$ are solutions of the Maxwell's equations in a transformed structure described by the transformed material matrix:

$$\bar{\mathbf{M}}_d = \det\left(\bar{\mathbf{S}}\right)\left(\bar{\mathbf{S}}^{-1}\right)^{\mathrm{T}} \cdot \bar{\mathbf{M}} \cdot \bar{\mathbf{S}}^{-1}, \tag{5}$$

where $\bar{\mathbf{M}}$ is the material matrix of a Tellegen medium defined as

$$\begin{pmatrix} \mathbf{D}/\varepsilon_0 \\ \mathbf{B}c \end{pmatrix} = \bar{\mathbf{M}} \cdot \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix}, \text{ with } \quad \bar{\mathbf{M}} = \begin{pmatrix} \varepsilon & \kappa \\ \kappa & \mu \end{pmatrix}. \tag{6}$$

From (5) it follows that $\det(\overline{\mathbf{M}}) = \det(\overline{\mathbf{M}}_d)$, with $\det(\overline{\mathbf{M}}) = \varepsilon\mu - \kappa^2 = n^2$ and $\det(\overline{\mathbf{M}}_d) = \varepsilon_d \mu_d - \kappa_d{}^2 = n_d{}^2$. Hence, $\overline{\mathbf{M}} \to \overline{\mathbf{M}}_d$ induces a mapping of the Riemann sphere into itself. Note that the coefficients of $\overline{\mathbf{S}}$ must be defined over the real numbers because we are only dealing with lossless media.

3.1 Classification of Duality Transformations

It is possible to develop a classification of the duality transformations based on the location of the respective fixed points on the Riemann sphere. A detailed analysis shows that a duality transformation induces a Möbius mapping in the complex plane:

$$w \to w_d = \left(s_{22}w - s_{21}\right) / \left(s_{11} - s_{12}w\right), \tag{7}$$

where $w = M + i\Gamma$ and $w_d = M_d + i\Gamma_d$ represent the original and the transformed Tellegen media in the (M, Γ) plane. From (7) it is clear that there are exactly two fixed points (i.e. solutions of $w = w_d$), and that the fixed points are either complex conjugated or real valued. In the former case one of the fixed points lies in the region $\Gamma > 0$ (Tellegen with positive refractive index), and the other one in the region $\Gamma < 0$ (Tellegen with negative refractive index), and in latter case the fixed points lie on the PEMC line ($\Gamma = 0$). Moreover, using the inverse of stereographic projection the fixed points can be geometrically projected onto the Riemann sphere corresponding to *i*) two "mirror" Tellegen media located in the hemispheres $y > 0$ and $y < 0$ or *ii*) two distinct points in the PEMC circle ($x^2 + z^2 = 1$ meridian) or *iii*) two coincident points in the PEMC circle. Interestingly, the duality transformations are conformal mappings in the Riemann sphere, and in particular transform circles into circles. We can prove this result by arguing that the composition of conformal mappings (Möbius transformation and stereographic projection) is still a conformal transformation.

Depending on the case (i), (ii) or (iii) described above, we classify a duality transformation as a generalized rotation, a generalized Lorentz boost or a generalized Galilean boost, respectively. A detailed analysis shows that the duality transformations with $\det\bar{\mathbf{S}}=1$ can be written in the exponential form as $\bar{\mathbf{S}}=e^{u\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}$, where u is some real-valued parameter, $\hat{\mathbf{n}}=\left(n_x,n_y,n_z\right)$ is a vector, and $\boldsymbol{\sigma}=\left(\boldsymbol{\sigma}_x,\boldsymbol{\sigma}_y,\boldsymbol{\sigma}_z\right)$ represent the Pauli matrices [21]. Moreover, it can be shown that for the generalized rotations $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}=-1$, for the Lorentz boosts $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}=1$, whereas the Galilean boosts have $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}=0$. Here, n_x,n_z are real-valued parameters and n_y is imaginary pure.

3.2 Geometrical Action of Duality Transformations

As mentioned previously, the three classes of duality transformations are characterized by the geometric location of their fixed points, that is, the media that are unaltered under a transformation.

The generalized rotations are characterized by $\bar{\mathbf{S}}_R=e^{(\theta/2)\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}$ with $u=\theta/2$ and require the following parameterization of the vector $\hat{\mathbf{n}}$ in terms of two arbitrary real-valued parameters (χ_R,ϕ_R)

$$\hat{\mathbf{n}}=\cos(\phi_R)\tan(\chi_R)\hat{\mathbf{x}}-i\sec(\chi_R)\hat{\mathbf{y}}-\sin(\phi_R)\tan(\chi_R)\hat{\mathbf{z}}\,. \tag{8}$$

It can be proven that (χ_R,ϕ_R) and $(\pi-\chi_R,\phi_R)$ determine the spherical coordinates of the two fixed the points of the transformation on the Riemann sphere [see Fig. 1b], and that θ can be identified with a rotation angle, such that the action of the duality transformation is to rotate the points of the Riemann sphere around the fixed point. In general, the rotation is not a standard Euclidean geometry rotation, and hence the name “generalized”.

On the other hand, the generalized Lorentz boosts $\bar{\mathbf{S}}_L=e^{u_L\,\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}$ have their two fixed points on the PEMC circle which are named *sink* and *source* and defined by the coordinates $(\chi=\pi/2,\phi_{L,si})$ and $(\chi=\pi/2,\phi_{L,so})$, respectively. When $u_L>0$ $\left(u_L<0\right)$ the points on the Riemann sphere, with the exception of the two fixed points, are dragged towards the sink (source). For these transformations, the vector $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \cos\left(\frac{\phi_{L,so}+\phi_{L,si}}{2}\right)\csc\left(\frac{\phi_{L,so}-\phi_{L,si}}{2}\right)\hat{\mathbf{x}} + -i\cot\left(\frac{\phi_{L,so}-\phi_{L,si}}{2}\right)\hat{\mathbf{y}} - \sin\left(\frac{\phi_{L,so}+\phi_{L,si}}{2}\right)\csc\left(\frac{\phi_{L,so}-\phi_{L,si}}{2}\right)\hat{\mathbf{z}} \,. \quad (9)$$

Finally, the generalized Galilean boost $\overline{\mathbf{S}}_G = e^{u_G \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}$ has two coincident fixed points on the PEMC circle defined by the arbitrary coordinates $(\chi = \pi/2, \phi_G)$. It is characterized by the vector

$$\hat{\mathbf{n}} = -\cos(\phi_G)\hat{\mathbf{x}} + i\hat{\mathbf{y}} + \sin(\phi_G)\hat{\mathbf{z}} \,. \quad (10)$$

The geometrical interpretation of each transformation type is illustrated in Fig. 2.

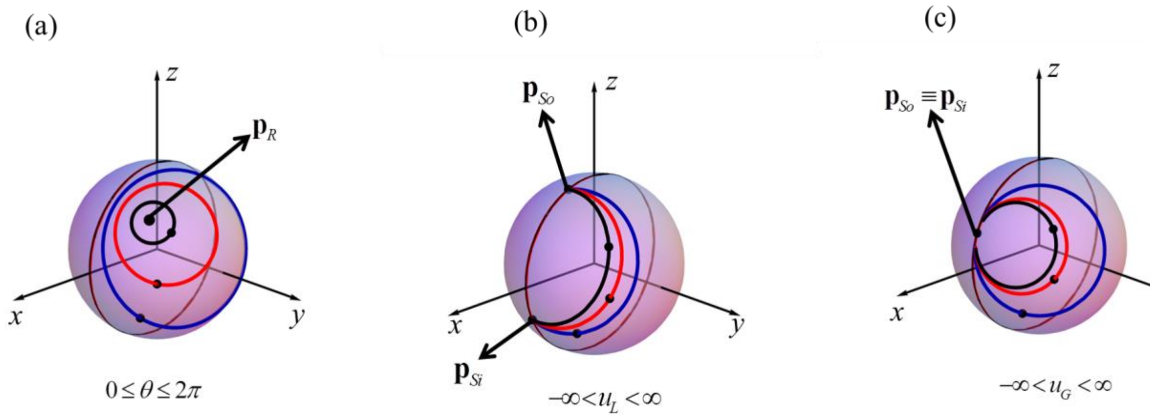

Fig. 2 Geometrical interpretation of the duality transformations. (a) Generalized rotation with the fixed point $\mathbf{p}_R$ defined by the spherical coordinates $(\chi_R, \phi_R) = (\pi/3, \pi/4)$. Trajectories generated by a family of rotations ($0 \le \theta \le 2\pi$) applied to selected points (the black dots). (b) Lorentz boost defined by the sink and source points, $\mathbf{p}_{Si}$ and $\mathbf{p}_{So}$, respectively, characterized by the spherical coordinates $(\phi_{L,so}, \phi_{L,si}) = (\pi/8, 5\pi/8)$. Trajectories generated by a family of Lorentz boosts ($-\infty \le u_L \le \infty$) applied to selected points. (c) Galilean boost with coincident fixed points, $\mathbf{p}_{Si} \equiv \mathbf{p}_{So}$, defined by $\phi_G = 3\pi/8$. Trajectories generated by a family of Galilean boosts ($-\infty \le u_G \le \infty$) applied to selected points.

4. The most general classes of Tellegen media reducible to simple media

In this section, the known theories that enable reducing one Tellegen medium to a simple isotropic medium [5-7] are generalized. We derive the most general sets of Tellegen media that are reducible to the SIM family under the same duality transformation. Each of these sets is named a *Tellegen class*.

Because duality transformations are conformal mappings, the SIM circle [see Fig. 1c] is taken by any duality mapping into another circle. Moreover, the duality transformed SIM circle is such that if a point lies in the circle then its "mirror image" point with respect to the y=0 plane also belongs to the circle. This demonstrates that the most general *Tellegen classes* that can be mapped into the SIM circle are represented in the Riemann sphere by circles invariant to reflections with respect to the y=0 plane. Because three points of a circle completely characterize it, we conclude that a *Tellegen class* is completely determined by any two points in the hemisphere y>0. In summary, all the Tellegen media, (irrespective of their refractive index), that are represented in the Riemann sphere by a circle invariant to reflections with respect to the y=0 plane are in the same Tellegen class. Moreover, it can be shown that three materials are in the same class if and only if their constitutive parameters are linearly dependent such that:

$$\det\begin{pmatrix} \varepsilon_1 & \mu_1 & \kappa_1 \\ \varepsilon_2 & \mu_2 & \kappa_2 \\ \varepsilon_3 & \mu_3 & \kappa_3 \end{pmatrix} = 0 . \quad (11)$$

Given any two points (Tel_1) and (Tel_2) in the hemisphere y>0 the associated Tellegen class can be taken into the SIM circle by a suitable duality transformation [see Fig. 3a]. In particular, this transformation allow us to transform (Tel_1) and (Tel_2), into the two corresponding SIMs, (SIM_1) and (SIM_2). Indeed, suppose that the two Tellegen media are characterized by the constitutive parameters $(\varepsilon_1, \mu_1, \kappa_1)$ and $(\varepsilon_2, \mu_2, \kappa_2)$, respectively, and that SIMs, (SIM_1) and (SIM_2), are defined by the constitutive parameters (ε_1', μ_1') and (ε_2', μ_2'), respectively. In a first step, we transform (Tel_1) into (SIM_1) using a generalized rotation $\bar{\mathbf{S}}_{R,1} = e^{(\theta_1/2)\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}$ that leaves the vacuum invariant (fixed point) and the rotation angle $\theta_1 = \arctan\left[2\kappa_1/(\mu_1 - \varepsilon_1)\right]$, as illustrated in Fig. 3bi. This rotation corresponds to a mapping defined by the coefficients $s_{11} = s_{22} = \cos(\theta_1/2)$ and $s_{12} = -s_{21} = -\sin(\theta_1/2)$. It can be checked in Fig. 3bi that the original media are exactly rotated by an angle θ_1 around the y-axis where the fixed point (the vacuum) is located. Under the action of this first transformation $\bar{\mathbf{S}}_{R,1}$ the Tellegen medium (Tel_2) is transformed into another Tellegen medium

(Tel_2') with the constitutive parameters $(\varepsilon_2', \mu_2', \kappa_2')$ [Fig. 3bi]. In a second step, we reduce (Tel_2') to (SIM_2) using a generalized rotation $\bar{\mathbf{S}}_{R,2} = e^{(\theta_2/2)\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}$ with fixed point equal to the (SIM_1), see Fig. 3bii. In this case, the rotation angle is $\theta_2 = \arctan\left[2\kappa_2'/\left(\mu_2'/\eta_1' - \varepsilon_2'\,\eta_1'\right)\right]$ where $\eta_1' = \sqrt{\mu_1'/\varepsilon_1'}$.

Evidently, the composition of two generalized rotations $\bar{\mathbf{S}}_{T-S} = \bar{\mathbf{S}}_{R,2} \cdot \bar{\mathbf{S}}_{R,1}$ reduces the two Tellegen media into two SIMs, as desired. In the example of Fig. 3a, (Tel_1) and (Tel_2) are determined by $(\chi_1 = \pi/4, \phi_1 = \pi/3)$ and $(\chi_2 = \pi/7, \phi_2 = \pi/4)$, where the spherical coordinates (χ, ϕ) are defined as in Fig. 1b. The corresponding constitutive parameters can be obtained through (3). In the simulations of Fig. 3b, the Tellegen *class* represented by the black curve, defined by (Tel_1) and (Tel_2) [see Fig. 3bi], is mapped by $\bar{\mathbf{S}}_{R,1}$ into another black curve containing (SIM_1) and (Tel_2'). Finally, using $\bar{\mathbf{S}}_{R,2}$, the transformed black curve of Fig. 3bi is taken to the SIM class, as illustrated in Fig. 3bii.

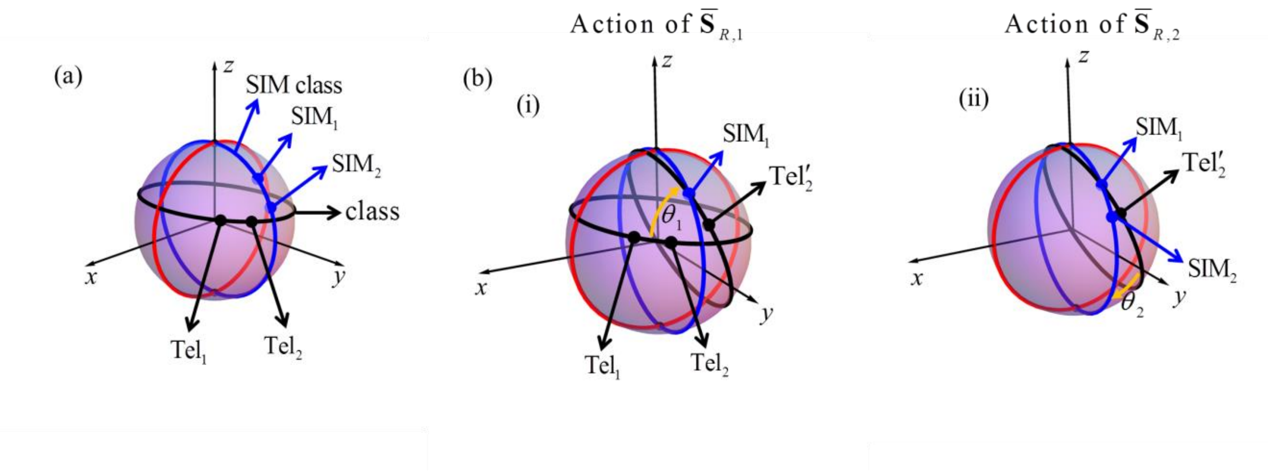

Fig. 3 Any two Tellegen media (Tel_1) and (Tel_2) in the hemisphere y>0 determine univocally a *class* of media that is reducible to SIMs. (blue circle). (a) Geometrical representation of a *class* of Tellegen media that can be transformed into the SIM class. (b) (i) The media represented by (Tel_1) and (Tel_2) are transformed into (SIM_1) and (Tel_2') under the transformation $\bar{\mathbf{S}}_{R,1}$. (ii) The Tellegen medium represented by (Tel_2') is taken to (SIM_2) by the transformation $\bar{\mathbf{S}}_{R,2}$, leaving (SIM_1) invariant.

5. Periodic structures

In this section, we demonstrate how the geometric interpretation of duality transformations on the Riemann sphere can be effectively used to solve wave propagation problems involving Tellegen media. We characterize the dispersion of the Bloch modes of photonic crystals formed by Tellegen media. In section 5.1, we consider the scenario wherein the crystal is formed by two different materials [Fig. 4a]. As previously discussed, in this case it is always possible to reduce the structure to a photonic crystal formed by reciprocal media. In section 5.2, we investigate Tellegen crystals such that the unit cell has three different materials [Fig. 4b]. In general such a structure is not amenable to be transformed into a reciprocal photonic crystal, unless by coincidence the three materials belong to a Tellegen class. As a consequence, we will see that the band diagrams of crystals with the unit cell of Fig. 4b are fundamentally different from those characteristic of reciprocal photonic crystals.

The operation of reducing Tellegen structures to SIM structures allows us to determine the electromagnetic fields of the original problem by applying the inverse duality transformation to the solution of the transformed problem. One remarkable thing is that duality transformations only act over the electromagnetic fields, leaving the spatial coordinates $\mathbf{r} = (x, y, z)$ and the time coordinate t invariant. As consequence, the band structure of a photonic crystal is invariant under a duality mapping, i.e the dispersion diagrams ω vs. $\mathbf{k}$ are precisely the same for the original crystal and for a duality-transformed crystal. Here, ω represents the frequency and $\mathbf{k}$ the wave vector of a generic Bloch mode. For example, the band structure of a one-dimensional Tellegen crystal involving two different Tellegen [see Fig. 4a] materials is exactly the same as that of the duality transformed crystal wherein the original materials are mapped into SIMs.

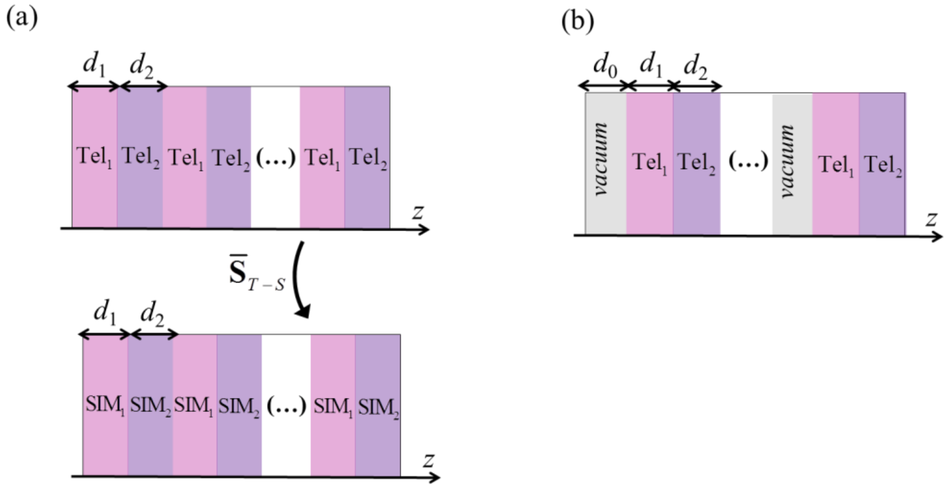

Fig. 4 Geometry of photonic crystals formed by Tellegen media. (a) A periodic structure formed by two different Tellegen materials, (Tel_1) and (Tel_2) is amenable to be reduced to a crystal with two SIMs, (SIM_1) and (SIM_2), under the action of the duality transformation $\bar{\mathbf{S}}_{T-S}$. (b) Photonic crystal formed by Tellegen media, (Tel_1) and (Tel_2), and the vacuum.

5.1 Tellegen Media in the Same Class

Here, we consider Tellegen periodic structures with the geometry of Fig. 4a. Each cell of the structure is formed by two Tellegen media ,(Tel_1) and (Tel_2), with thicknesses d_1 and d_2. The period is $a = d_1 + d_2$. For simplicity, we restrict our attention to the case wherein the propagation is along the *z*-direction.

Using the previously discussed duality transformation $\bar{\mathbf{S}}_{T-S}$, the Tellegen photonic crystal can be transformed into a SIM photonic crystal formed by (SIM_1) and (SIM_2). This operation leaves the thicknesses d_1 and d_2 and the *z*-propagation constants $\beta_1 = n_1\omega / c$ and $\beta_2 = n_2\omega / c$ invariant. It is well known that the dispersion equation for the Bloch waves of the SIM photonic crystal is given by

$$\cos(k_z a) = \cos(\beta_1 d_1)\cos(\beta_2 d_2) - \left[\frac{\beta_1^2 \mu_2' + \beta_2^2 \mu_1'}{\beta_2\ \beta_1\ \mu_1'\ \mu_2'}\right]\frac{\sin(\beta_1 d_1)\sin(\beta_2 d_2)}{2}, \qquad (12)$$

being k_z the propagation constant of the Bloch wave. Note that for propagation along the *z*-direction ($k_x = k_y = 0$) the Bloch waves are degenerate because the TE and TM waves have the same characteristic dispersion. According to our theory, the band diagram of the Tellegen periodic structure can be calculated from (12). The result of such a calculation is illustrated in Fig. 5a, for Tellegen media (Tel_1) and (Tel_2) defined by the material parameters $(\varepsilon_1 = 3, \mu_1 = 2, \kappa_1 = 1.5)$ and $(\varepsilon_2 = 1, \mu_2 = 3, \kappa_2 = 1)$, respectively. These media are transformed through $\bar{\mathbf{S}}_{T-S}$ to (SIM_1) and (SIM_2) with the constitutive parameters $(\varepsilon_1' = 2.03, \mu_1' = 1.85)$ and $(\varepsilon_2' = 0.59, \mu_2' = 3.41)$, respectively.

To validate our analysis we calculated independently the band structure of the original Tellegen photonic crystal. This requires solving a more difficult problem wherein all the involved materials have magnetoelectric coupling. To do this we make an analysis based on scattering matrices [24]. Indeed, it is known that the dispersion equation of a Tellegen periodic structure can be expressed as:

$$\det\left(\begin{bmatrix} e^{-ik_z a}\,\bar{\mathbf{T}}^L - \bar{\mathbf{I}} & \bar{\mathbf{R}}^R \\ \bar{\mathbf{R}}^L & e^{ik_z a}\,\bar{\mathbf{T}}^R - \bar{\mathbf{I}} \end{bmatrix}\right) = 0, \qquad (13)$$

where $\bar{\mathbf{I}}$ is the 2x2 identity unitary matrix. In the above, the matrices $(\bar{\mathbf{R}}^L, \bar{\mathbf{T}}^L)$ and $(\bar{\mathbf{R}}^R, \bar{\mathbf{T}}^R)$ represent the reflection and transmission matrices for the transverse components of the electric field calculated for plane wave normal incidence on a single cell of the structure standing alone in a vacuum. The superscripts “R” and “L” indicate if the incoming wave comes from the left (propagates along the +z direction) and or from the right (propagates along the –z direction) hand side. Note that typically $(\bar{\mathbf{R}}, \bar{\mathbf{T}})$ are not scalars because Tellegen media originate polarization conversion. The reflection and transmission matrices are computed based on standard (plane wave) modal expansions and mode matching at the interfaces.

In the Fig. 5a, we plot the band diagram of the Tellegen photonic crystal obtained from the solution of equation (13) using $d_1 = d_2 = 0.5a$. As expected, the results are exactly coincident with what is obtained using duality theory. Moreover, as previously mentioned, each line is associated with two distinct modes because of the polarization degeneracy. Thus, for Tellegen media in the same *class* the Bloch modes are doubly degenerated.

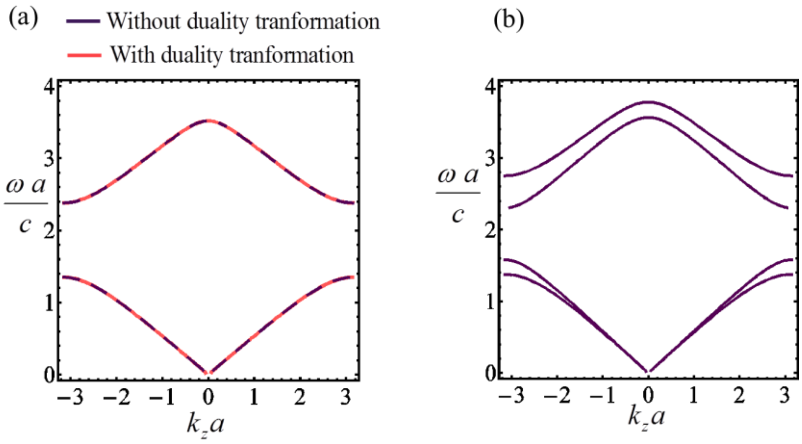

Fig. 5 Band diagrams of Tellegen periodic structures (a) Unit cell with media in the same *class.* (b) Unit cell with media irreducible to the SIM *class.*

5.2 Irreducible Tellegen Media

It is interesting to investigate the band diagrams of Tellegen photonic crystals that are irreducible to SIM photonic crystals. It is evident that we need to consider crystals formed by at least three different materials [Fig. 4b]. As an example, we consider the combination of (Tel_1), (Tel_2) and the vacuum, and choose the parameters of the Tellegen materials such that the three materials cannot be simultaneously reduced to SIMs. The band diagram of this photonic crystal can be calculated again based on the dispersion equation (13), considering now that $a = d_0 + d_1 + d_2$, being (d_1, d_2, d_0) the thicknesses of the pertinent regions. The calculated band diagram for the case $(d_1, d_2, d_0) = (0.4, 0.4, 0.2)a$ is depicted in Fig. 5b. The constitutive parameters of the Tellegen media are the same as in the example of Fig. 5a. Notably, the numerical results show that when the Tellegen photonic crystal cannot be reduced to a SIM photonic crystal the Bloch modes are no longer degenerate, and thus we have two distinct dispersion curves. Moreover, a detailed analysis reveals the corresponding Bloch eigenmodes are circularly polarized. This example demonstrates unequivocally that the electrodynamics of Tellegen photonic crystals can be fundamentally different from that of conventional media, and reveals the distinct signature of Tellegen media in wave propagation problems.

6. Conclusion

We developed a geometrical interpretation of duality transformations for Tellegen media in the Riemann sphere. It was shown that each family of isorefractive Tellegen media can be identified with the unit sphere. A classification of duality transformations has surfaced naturally from the geometrical action of the transformations in the Riemann sphere. The duality mappings were classified into generalized rotations, Lorentz boosts and Galilean boosts. The fixed points of the transformations can be either Tellegen media or electromagnetic boundaries depending on the type of transformation. Duality transformations are conformal mappings being this result another important outcome our geometrical theory.

We demonstrated that these geometric ideas lead to a clearer understanding of the opportunities created by duality transformations in the reduction of Tellegen media to simpler media. Specifically, we have shown that any two arbitrary Tellegen materials can always be reduced to SIMs under a composition of two generalized rotations. In addition, Lorentz and Galilean boosts can be quite useful in problems wherein it is desired to fix an electromagnetic boundary (e.g. a PEC boundary) and transform a Tellegen material into a SIM. Moreover, the most general classes of Tellegen media reducible to SIMs were geometrically and analytically characterized. It was shown that they are represented by circles invariant to reflections with respect to the y=0 plane in the Riemann sphere.

In particular, our theory was applied to the study of wave propagation in Tellegen photonic crystals, and it was demonstrated how some complex structures can be easily handled with duality transformations. It was shown that the band structure of Tellegen photonic crystals stays invariant under a duality transformation. In particular, the band structure of Tellegen photonic crystals formed by media in the same class can be found from the band structure of conventional SIM photonic crystals. We also investigated the band structures of Tellegen crystals irreducible to SIMs, proving that they are topologically distinct from those of conventional media. This reveals the distinctive and unique signature of Tellegen materials in wave propagation problems. To conclude, we would like to note that our theory can be applied to a wide range of propagation and radiation problems.

7. References

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