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Dynamic Debt With Intensity-Based Models

João Miguel Reis[1,2] | José Carlos Dias[1,3]

[1]Business Research Unit (BRU-IUL), Lisbon, Portugal | [2]Universidade Europeia, Lisbon, Portugal | [3]Instituto Universitário de Lisboa (ISCTE-IUL), Lisbon, Portugal

Correspondence: João Miguel Reis (joao.reis@universidadeeuropeia.pt)

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ABSTRACT

This article proposes a dynamic debt model where the face value of debt can change. In particular, our dynamic debt setting allows debt changes ruled by intensity processes that are linked to the firm value through the correlation between the stochastic processes. Analytical solutions are obtained, and we extend the proposed dynamic debt model to the case of subordinated debt. While empirical behaviors are emulated, the impacts of dynamic debt over the credit spreads are explored. In this model, the possibility of debt increases magnifies credit spreads and the reverse occurs for the possibility of debt decreases.

JEL Classification: G12, G32

1 | Introduction

The pricing of risky debt claims and corresponding credit spreads constitutes a rich field of research. The structural default models approach begins with Black and Scholes (1973) and Merton (1974), where a fixed debt level is held and, at the debt's maturity date, the total value of the firm determines if a default event is triggered.

To deal with the limitations that arise from its simplified assumptions, various extensions to the baseline model of Merton (1974) have been proposed. Here, the challenged assumption is that of firms holding fixed amounts of debt; contrary to what is assumed by the baseline model, firms tend to adjust debt over time. One can expect more debt to be accrued when the value of the firm increases and less debt when it decreases. For a debt holder, the future face value of the total debt held by the firm can impact not only the probability of default but also the share to be received in the event of default.

Several studies point to the value of analyzing adjustments in the debt levels. Roberts and Sufi (2009) mention that most of long-term debt contracts suffer renegotiations over the amount, maturity, and pricing of the contract. Nini et al. (2012) find that creditors, through informal channels, play an active role in the governance of firms, even when default is a far scenario.

Crucially, there are empirical results that establish a link between credit spreads and expected leverage variations. In a transversal analysis of the Merton model, Gemmill (2002) points to the importance of considering downward changes in leverage to solve the problem of the counterfactual downward slopping term-structure for riskier bonds. In Flannery et al. (2012), by analyzing the bonds of nearly unique 400 firms, it is shown that expected future leverage affects bond yields, above and beyond the effects of the observed leverage, implying that “leverage expectations are the dominant influence on a firm's credit spread.” In addition, the effect is not systematically different among expected increases and decreases in leverage. More recently, Maglione (2022), through a cointegration approach involving credit spreads and leverage (plus equity volatility), finds evidence of a long-term relation among these variables. This suggests that spreads are not solely priced in relation to the current leverage, but are also tied to the values to which leverage (plus equity volatility) is expected to converge.

Several attempts have been made to introduce a dynamic behavior on the level of debt. For instance, in the Collin-Dufresne and Goldstein (2001) setting, debt changes continuously, with a process for the debt level that is linked to the firm value and is steered toward a defined leverage. However, the model relies on continuous changes in the debt model and a strict adherence to a given capital structure.

The assumption about the strictness of the capital structure is disputed. For instance, DeAngelo and Roll (2015) find that capital structure stability is an exception. In addition, the survey responses in Graham and Harvey (2001) regarding the leverage policy of firms indicate that while 19% do not have a target and 10% have a strict target, the remaining majority have either a flexible target or a somewhat tight target or range.

Das and Kim (2015) set a model where this limitation is addressed by allowing adjustments in the leverage after substantial deviations from the initial value. The debt level can increase or decrease when certain barriers are crossed by the firm value. By setting the knock-in and knock-out values for the barrier options—to indicate the firm values at which the debt amounts are changed—and adjusting the strike prices properly—to indicate how much debt changes—Das and Kim (2015) are able to simulate debt ratchets and write-downs in a discrete fashion, where there is a free movement of the leverage within a range, and once barriers are crossed, the face value of debt (and, therefore, the leverage) is adjusted. Consequently, after sizeable decreases in the leverage due to firm value increases, the debt level increases (thus increasing the leverage), and the reverse occurs when the firm value decreases. They study a total of six cases of possible debt changes, which are compared with the baseline Merton (1974) model.

More recently, Eisenthal-Berkovitz et al. (2020) explore the dynamics of leveraged buyout (hereafter, LBO) events and explore a structural model case where the firm can suffer an LBO, which triggers an increase in the debt level. To do so, a Cox process attributes a probability to the event of the LBO, which is defined exogenously and is independent of variations in the firm value.

Our paper adopts the dynamic debt mechanism of Das and Kim (2015), but avoids the need to initially specify the particular constant barriers that trigger the increases and decreases in the nominal debt. To do so, we use hazard processes to trigger the debt change events. More specifically, intensity processes that can be exogenous are used, although there is the possibility of including correlation between a process and the firm's value, thus allowing variations in the firm value to influence the direction of the debt changes. Therefore, while avoiding setting precise debt ratios to trigger the debt changes and not assuming a strict leverage policy, we are able to retain the Das and Kim (2015) feature of debt being changed, whilst tending to increase it when the firm value increases and decrease it when the firm value decreases. Since Black and Cox (1976), various approaches to credit risk with barrier options have emerged, such as Brockman and Turtle (2003) and Lin and Sun (2009).

By using a Vasicek (1977) intensity process, this model opens the possibility to calibrate the debt changes under its own process, and the correlation parameterization allows one to approach different degrees of strictness in terms of leverage targeting. Hence, this model structure allows investors to study debt values and spreads under the possibility of a dynamic capital structure without the requirement of explicitly choosing the ex ante levels at which the debt changes are triggered.

As for the arguments presented in Das and Kim (2015) to justify the possibility of debt changes alongside changes in the firm value, it is argued that, in the case of the leverage increase, as the firm increases in value, the extra collateral can be used to support the additional debt; as for the debt decrease case, as the firm value decreases, to stave off default, the nominal debt amount is reduced. The arguments above can be complemented with a stream of literature that links firm value increases—see, for instance, Mura and Marchica (2010) and Denis and McKeon (2012)—and decreases—see, for example, DeAngelo et al. (2018) and Lambrinoudakis et al. (2019)—to adjustments in the nominal debt.

This kind of intensity process is not a novelty in the credit risk literature. Often, these are used to allow exogenous factors beyond the firm value to predict the default event, allowing the use of econometric specifications from term-structure modeling. Both Jarrow and Turnbull (1995) and Madan and Unal (1998) provide early examples of this approach when modeling two sources of risk simultaneously. In addition, the pricing of vulnerable options—contracts where, in addition to the usual risk of the asset price, the issuers' default risk is also taken into consideration—often relies on intensity processes. Klein (1996) and Klein and Inglis (2001) offer early examples of this approach. Recently, Fard (2015) and Koo and Kim (2017) provide studies where, to simulate the occurrence of default events, intensity-based models are used in vulnerable options. Baule (2021) provides a method to evaluate vulnerable derivatives, where the correlation between the underlying asset and the issuer is estimated through a simple method. In the context of default risk, Denault et al. (2009) compare different alternatives for processes used to estimate physical intensity processes.

Besides offering flexibility in regard to the strictness of the leverage targeting, the framework presented in this paper is adaptable to other existing structural models. Apart from studying the spreads on increases over the baseline Merton (1974) model, the possibility of a debt increase is also studied in the case of the presence of subordinated debt. Black and Cox (1976) present a model with multiple debt claimants. Afterward, Gorton and Santomero (1990) present a model where debt is separated between senior debt and junior debt, with the latter being impacted by the amount of the former. They reach a formula that can be interpreted as the difference between two call options, which can be adapted to the possibility of debt changes. Here, following that framework, we set the possibility of increasing the senior debt, and study how it impacts the credit spreads of the junior debt.

The remainder of this paper is organized as follows. In Section 2, we present the extensions to the Merton (1974) model with the possibilities of increasing and decreasing the debt and also approach the subordinated debt using the framework of Gorton and Santomero (1990). In Section 3, we obtain the closed-form solutions for the bonds. In Section 4, we provide some numerical results and explore the impact of different variables within the model. Finally, in Section 5, we present the conclusions.

2 | The Random Dynamic Debt Model

2.1 | The Baseline Merton Model

We begin by introducing the Merton (1974) model, which is the baseline for this study. For a given firm, the face value of debt is represented by D and the maturity of the zero-coupon bond is T. The total firm value (i.e., the sum of the market value of equity and the market value of debt) is denoted by V and is assumed to follow the usual (risk-neutral) geometric Brownian motion, that is,

$$\frac{dV_t}{V_t} = (r - q)dt + \sigma_V dW_{t,V}^{\mathbb{Q}}, \tag{1}$$

where the risk-free rate is represented by r, q stands for the firm's total payout to debt and equity holders, and σ_V represents the standard deviation of the firm value. The known solution is

$$V_t = V_0 \exp\left[\left(r - q - \frac{1}{2}\sigma_V^2\right)t + \sigma_V W_{t,V}^{\mathbb{Q}}\right], \tag{2}$$

with $W_{t,V}^{\mathbb{Q}}$ being a standard Brownian motion defined under the measure $\mathbb{Q}$ and generating the filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$.

In this model, the firm issues a zero-coupon bond. Hence, the default event can only occur at maturity and does so when the firm's value is below the face value of debt, that is, when $V_T < D$. When default occurs, the debt holder obtains a fraction of the firm's value $\phi_{\text{dwl}} V_T$. The inclusion of $\phi_{\text{dwl}} (\leq 1)$ contemplates the possibility of a deadweight default loss, that is, the possibility of not recovering the full firm's value upon default, but only the fraction ϕ_{dwl}. In the case of default not occurring, at maturity, the debt holder receives the face value of the debt D.

Following the standard solution, the time-0 bond value is given by

$$\begin{aligned} B_0(V_0, D, T) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \leq D\}} \middle| \mathcal{F}_0\right] \\ &= De^{-rT}N(d_2(V_0, D, T)) \\ &\quad + \phi_{\text{dwl}} V_0 e^{-qT} N(-d_1(V_0, D, T)), \end{aligned} \tag{3}$$

where

$$d_{1,2}(V_0, D, T) = \frac{\ln(V_0/D) + \left(r - q \pm \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}. \tag{4}$$

Finally, we recall that the corresponding time-0 equity value in this modeling setting is given by

$$\begin{aligned} E_0(V_0, D, T) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(\phi_{\text{dwl}} V_T - D)\mathbb{1}_{\{V_T > D\}} | \mathcal{F}_0\right] \\ &= \phi_{\text{dwl}} V_0 e^{-qT} N(d_1(V_0, D, T)) \\ &\quad - De^{-rT}N(d_2(V_0, D, T)). \end{aligned} \tag{5}$$

2.2 | Debt With the Possibility of an Increase—Debt Ratchet

Now, we establish the mechanism for the leverage increase through a rise in the nominal debt, that is, a debt ratchet. Our objective goes beyond allowing the nominal debt value (and, therefore, the leverage) to increase; we also aim to connect that possibility to the path of the firm value. As the firm value increases through the diffusion process, the leverage is decreased (for the fixed amount of nominal debt) and this gives a wider margin to the management of the firm to increase the debt amount. Therefore, as the leverage is decreased by market events over the (increased) firm value, a debt ratchet event becomes more likely, and when it occurs, this will result in a debt increase (and increased leverage as a result).

We do not point to firms having a strict target for the leverage, although, after being decreased due to a greater firm value, with the presented dynamic, we foresee it to revert to higher levels as a result of the increased debt from the ratchet event. On the other side, decreases in the firm value reduce the likelihood of the debt ratchet increasing the nominal debt amount.

As for empirical studies where this relation is explored, Mura and Marchica (2010) find that firms maintain a low leverage to have flexibility for large investments to be financed through new debt—justifying the expectation of potential increases in the nominal debt for high-grade debt firms—and Denis and McKeon (2012) argue that investment related capital needs—which one can relate to an increase in the firm value—determine the debt level decisions.

Here, we assume the possibility of one increase in the nominal value of debt, which can occur between time-0 and the maturity date T. The inclusion of a correlation parameter, between the firm value and the possibility of the debt increase processes, will allow one to link the probability of a debt increase to greater firm values.

Following the same mechanism as in Das and Kim (2015), the increase in the nominal debt that augments the leverage is corresponded by a decrease in the equity, holding the total firm value equal. That is, at the moment of the debt increase event, the increased amount of debt to be reimbursed at maturity corresponds to a new bond value, and that value is subtracted from the equity. We also assume no restructuring costs. This mirrors how the debt decrease is approached in Section 2.3.

The debt ratchet event is assumed to follow a Vasicek (1977) process with intensity λ_t^u, that is, for a small time interval Δ, the probability of a ratchet event occurring between t and $t + \Delta$ is approximately $\lambda_t^u \Delta$.

Following Lando (1998, eqs. 2.4 and 2.2), respectively, we note that $P(\tau_u > T) = \mathbb{E}\left[e^{-\int_0^T \lambda_t^u dt}\right]$ and $\tau_u = \inf\left\{\int_0^t \lambda(V_t)\,dt \geq E\right\}$, where E is a unit exponential random variable. The Vasicek (1977) process is given by

$$d\lambda_t^u = \kappa_u\left(\theta_u - \lambda_t^u\right)dt + \sigma_u dW_{t,u}^{\mathbb{Q}}, \tag{6}$$

with θ_u being the risk-adjusted long-term average of the process, κ_u the speed of the reversion, and σ_u the volatility. The correlation between the debt increase and the firm value is defined as $dW_{t,u}^{\mathbb{Q}} dW_{t,V}^{\mathbb{Q}} = \rho_{uV}\,dt$. By selecting $\rho_{uV} > 0$, the firm value and the likelihood of increasing the debt level become positively correlated, as intended.

The known unique solution to the stochastic differential Equation (6) is given by[1]

$$\lambda_t^u = \lambda_0^u e^{-\kappa_u t} + \theta_u(1 - e^{-\kappa_u t}) + \sigma_u \int_0^t e^{-\kappa_u(t-s)}\,dW_{s,u}^{\mathbb{Q}}. \tag{7}$$

The total firm debt level starts at D, and, if no ratchet occurs, that is, if $\tau_U > T$, it remains at that level. Thus, in this case, we have the same payoff as in the baseline Merton (1974) model, that is,

$$B_T^U(V_T, D, T) = D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \leq D\}}. \tag{8}$$

If the ratchet event occurs, with $U > 0$, the debt level is increased to De^U. Assuming the same maturity date and the same seniority among the new and the old debt, the indicator functions are adjusted to reflect the new amount of nominal debt. The amount to be received in the case of default—$V_T \leq De^U$—is weighted by $\frac{D}{De^U} = e^{-U}$, that is, the original debt holder will receive a percentage of the recovered V_T, equal to his share of the new total amount of debt (before considering the deadweight loss). The amount to be received when there is no default remains at D, as the initial debt holder's nominal debt is unchanged.

Hence, if $\tau_U \leq T$,

$$B_T^U(V_T, D, T) = D\mathbb{1}_{\{V_T > De^U\}} + \phi_{\text{dwl}} e^{-U} V_T \mathbb{1}_{\{V_T \leq De^U\}}. \tag{9}$$

The event of debt increase is assumed to be able to occur only after time-0 and $\mathbb{D} := \{\mathcal{D}_t, t \geq 0\}$ denotes the filtration generated by the indicator process $\mathcal{D}_t := \mathbb{1}_{\{t > \tau_U\}}$. In addition, $\mathbb{G} := \{\mathcal{G}_t : t \geq 0\}$ will denote the enlarged filtration obtained as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.

Therefore, the debt value at time-0 will be given by

$$\begin{aligned} B_0^U(V_0, D, T, \lambda_u, U) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[&\left(D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \leq D\}}\right)\mathbb{1}_{\{\tau_U > T\}} \\ &+ \left(D\mathbb{1}_{\{V_T > De^U\}} + \phi_{\text{dwl}} e^{-U} V_T \mathbb{1}_{\{V_T \leq De^U\}}\right) \\ &\mathbb{1}_{\{\tau_U \leq T\}} | \mathcal{G}_0\Big]. \end{aligned} \tag{10}$$

The above equation can be rewritten as

$$B_0^U(V_0, D, T, \lambda_u, U) = e^{-rT}\left(A_1^U + A_2^U + B_1^U - B_2^U + B_3^U - B_4^U\right), \tag{11}$$

where the first set of terms is

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[\left(D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \leq D\}}\right)\mathbb{1}_{\{\tau_U > T\}} |\, \mathcal{G}_0\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\left(D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \leq D\}}\right) e^{-\int_0^T \lambda_t^u dt} \middle|\, \mathcal{F}_0\right] \\ &= \underbrace{\mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > D\}} e^{-\int_0^T \lambda_t^u dt} \middle|\, \mathcal{F}_0\right]}_{A_1^U} \\ &\quad + \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \leq D\}} e^{-\int_0^T \lambda_t^u dt} \middle|\, \mathcal{F}_0\right]}_{A_2^U}, \end{aligned} \tag{12}$$

whereas the second set is

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[\left(D\mathbb{1}_{\{V_T > De^U\}} + \phi_{\text{dwl}} e^{-U} V_T \mathbb{1}_{\{V_T \leq De^U\}}\right)\mathbb{1}_{\{\tau_U \leq T\}} \middle|\, \mathcal{G}_0\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\left(D\mathbb{1}_{\{V_T > De^U\}} + \phi_{\text{dwl}} e^{-U} V_T \mathbb{1}_{\{V_T \leq De^U\}}\right)\left(1 - e^{-\int_0^T \lambda_t^u dt}\right) \middle|\, \mathcal{F}_0\right] \\ &= \underbrace{\mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > De^U\}} | \mathcal{F}_0\right]}_{B_1^U} - \underbrace{\mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > De^U\}} e^{-\int_0^T \lambda_t^u dt} \middle|\, \mathcal{F}_0\right]}_{B_2^U} \\ &\quad + \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} e^{-U} V_T \mathbb{1}_{\{V_T \leq De^U\}} | \mathcal{F}_0\right]}_{B_3^U} \\ &\quad - \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} e^{-U} V_T \mathbb{1}_{\{V_T \leq De^U\}} e^{-\int_0^T \lambda_t^u dt} \middle|\, \mathcal{F}_0\right]}_{B_4^U}. \end{aligned} \tag{13}$$

2.3 | Debt With the Possibility of a Decrease—Debt Write-Down

Now, we establish the mechanism of the debt decrease, that is, a debt write-down. As the firm decreases in value through its diffusion process, while the nominal amount of the debt remains the same, the leverage is increased. This imposes constraints on the management, increasing the incentive to decrease the nominal debt (and thus the leverage). Therefore, as the leverage is increased through market events, the more likely is for it to suffer a decrease via a debt write-down.

In the reverse, as the firm value increases (and the leverage is decreased), it becomes less likely for the nominal value of the debt to be decreased through the write-down.

These decreases find empirical backing in DeAngelo et al. (2018), who find a strong tendency for firms to deleverage from high levels of leverage to restore the option to borrow in the future—supporting expectations for nominal debt decreases in low-grade debt—and in Lambrinoudakis et al. (2019), who conclude that shocks on a firm's investment opportunity set—which one can interpret in a reduction of the firm value—tend to decrease the firm's leverage, especially on financially constrained ones.

Once again, this decrease in the amount of debt can occur between time-0 and time-T. Here, again, a correlation parameter cements this dynamic, that is, debt decreases are more likely after the firm looses value.

As in the debt increase case, we follow the mechanism of Das and Kim (2015), holding the firm value constant at the moment of the debt change. So, the nominal debt that was suppressed from the total debt has a value at the time of the debt decrease, and that value will increase the equity. We also assume that there are no restructuring costs.

Again, we use a Vasicek (1977) process, now with intensity λ_t^l, which follows the same dynamic

$$d\lambda_t^l = \kappa_l\left(\theta_l - \lambda_t^l\right)dt + \sigma_l dW_{t,l}^{\mathbb{Q}}, \tag{14}$$

with θ_l being the risk-adjusted long-term average of the process, κ_l the speed of the reversion, and σ_l the volatility. The correlation with the firm value is again present and defined as $dW_{t,l}^{\mathbb{Q}} dW_{t,V}^{\mathbb{Q}} = \rho_{lV}\,dt$. Now, with $\rho_{lV} < 0$, as the firm value decreases, the debt reduction becomes more likely.

As before, the known unique solution to the stochastic differential equation (14) is given by

$$\lambda_t^l = \lambda_0^l e^{-\kappa_l t} + \theta_l(1 - e^{-\kappa_l t}) + \sigma_l \int_0^t e^{-\kappa_l(t-s)}\, dW_{s,l}^{\mathbb{Q}}. \tag{15}$$

Once again, when no debt write-down is triggered, that is, if $\tau_L > T$, the nominal amount of debt remains at D and, therefore, the payoff is given by

$$B_T^L(V_T, D, T) = D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \le D\}}. \tag{16}$$

If the debt write-down occurs, that is, when $\tau_L \le T$, the debt level becomes De^L, with $L < 0$, while the assumptions on debt maturity again imply that, in case of default, the value received is weighted by $\frac{D}{De^L} = e^{-L}$, while in the case of no default, the amount to be received is unchanged.

Thus, in the cases where $\tau_L \le T$, the payoff to the original debt holder is

$$B_T^L(V_T, D, T) = D\mathbb{1}_{\{V_T > De^L\}} + \phi_{\text{dwl}} e^{-L} V_T \mathbb{1}_{\{V_T \le De^L\}}. \tag{17}$$

Hence, with the possibility of a debt decrease, the debt value will be

$$\begin{aligned} B_0^L(V_0, D, T, \lambda_l, L) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[&\left(D\mathbb{1}_{\{V_T > D\}} + \phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \le D\}}\right)\mathbb{1}_{\{\tau_L > T\}} \\ &+ \left(D\mathbb{1}_{\{V_T > De^L\}} + \phi_{\text{dwl}} e^{-L} V_T \mathbb{1}_{\{V_T \le De^L\}}\right)\mathbb{1}_{\{\tau_L \le T\}}\Big|\, \mathcal{G}_0\Big], \end{aligned} \tag{18}$$

which can be rewritten in similar terms to those of the debt increase, that is,

$$\begin{aligned} B_0^L(V_0, D, T, \lambda_l, L) = e^{-rT}\Big(&A_1^L + A_2^L + B_1^L - B_2^L \\ &+ B_3^L - B_4^L\Big), \end{aligned} \tag{19}$$

with

$$\begin{aligned} A_1^L &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > D\}} e^{-\int_0^T \lambda_t^l dt}\,\middle|\, \mathcal{F}_0\right], \\ A_2^L &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \le D\}} e^{-\int_0^T \lambda_t^l dt}\,\middle|\, \mathcal{F}_0\right], \\ B_1^L &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > De^L\}} | \mathcal{F}_0\right], \\ B_2^L &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T > De^L\}} e^{-\int_0^T \lambda_t^l dt}\,\middle|\, \mathcal{F}_0\right], \quad \text{and} \\ B_3^L &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} e^{-L} V_T \mathbb{1}_{\{V_T \le De^L\}} | \mathcal{F}_0\right], \\ B_4^L &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} e^{-L} V_T \mathbb{1}_{\{V_T \le De^L\}} e^{-\int_0^T \lambda_t^l dt}\,\middle|\, \mathcal{F}_0\right]. \end{aligned}$$

2.4 | Debt With the Possibility of an Increase and a Decrease

Now, the two previous cases are combined: the firm can have its value increased, decreased or both. Any of the events can occur, with the probabilities being ruled by Equations (6) and (14), plus the respective solutions. The order of the debt changes is not relevant, as the decisive factor is, if by the maturity date, the face value of debt has increased and/or decreased, how much of it is standing, and if the firm value can match it. There are four possible debt events: (i) no debt changes, with the total debt being D; (ii) debt increase, but no debt decrease, with the debt being De^U; (iii) debt decrease, but no debt increase, with the debt being De^L; and (iv) both debt increase and debt decrease, with the debt being De^Ue^L.

Alongside the correlations between the debt change processes and the firm value, it is also possible to include the correlation between the debt increase and decrease processes: $dW^{\mathbb{Q}}_{t,u}dW^{\mathbb{Q}}_{t,l} = \rho_{ul}dt$. Hence, the debt value will be given by

$$
\begin{aligned}
&B_0^{UL}(V_0, D, T, \lambda_u, U, \lambda_l, L)\\
&= e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[\Big(D\mathbb{1}_{\{V_T>D\}} + \phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\le D\}}\Big)\mathbb{1}_{\{\tau_U>T\}}\mathbb{1}_{\{\tau_L>T\}}\\
&\quad + \Big(D\mathbb{1}_{\{V_T>De^L\}} + \phi_{\text{dwl}}e^{-L}V_T\mathbb{1}_{\{V_T\le De^L\}}\Big)\mathbb{1}_{\{\tau_U>T\}}\mathbb{1}_{\{\tau_L\le T\}}\\
&\quad + \Big(D\mathbb{1}_{\{V_T>De^U\}} + \phi_{\text{dwl}}e^{-U}V_T\mathbb{1}_{\{V_T\le De^U\}}\Big)\mathbb{1}_{\{\tau_U\le T\}}\mathbb{1}_{\{\tau_L>T\}}\\
&\quad + \Big(D\mathbb{1}_{\{V_T>De^Ue^L\}} + \phi_{\text{dwl}}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T\le De^Ue^L\}}\Big)\\
&\quad \mathbb{1}_{\{\tau_U\le T\}}\mathbb{1}_{\{\tau_L\le T\}}|\mathcal{G}_0\Big]\\
&= e^{-rT}\Bigg\{\mathbb{E}_{\mathbb{Q}}\Bigg[\Big(D\mathbb{1}_{\{V_T>D\}} + \phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\le D\}}\Big)\\
&\quad e^{-\int_0^T\lambda_t^u dt}e^{-\int_0^T\lambda_t^l dt}\Bigg|\ \mathcal{F}_0\Bigg]\\
&\quad + \mathbb{E}_{\mathbb{Q}}\Bigg[\Big(D\mathbb{1}_{\{V_T>De^L\}} + \phi_{\text{dwl}}e^{-L}V_T\mathbb{1}_{\{V_T\le De^L\}}\Big)\\
&\quad e^{-\int_0^T\lambda_t^u dt}\Big(1 - e^{-\int_0^T\lambda_t^l dt}\Big)\Bigg|\ \mathcal{F}_0\Bigg]\\
&\quad + \mathbb{E}_{\mathbb{Q}}\Bigg[\Big(D\mathbb{1}_{\{V_T>De^U\}} + \phi_{\text{dwl}}e^{-U}V_T\mathbb{1}_{\{V_T\le De^U\}}\Big)\\
&\quad \Big(1 - e^{-\int_0^T\lambda_t^u dt}\Big)e^{-\int_0^T\lambda_t^l dt}\Bigg|\ \mathcal{F}_0\Bigg]\\
&\quad + \mathbb{E}_{\mathbb{Q}}\Bigg[\Big(D\mathbb{1}_{\{V_T>De^Ue^L\}} + \phi_{\text{dwl}}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T\le De^Ue^L\}}\Big)\\
&\quad \times\Big(1 - e^{-\int_0^T\lambda_t^u dt}\Big)\Big(1 - e^{-\int_0^T\lambda_t^l dt}\Big)\Bigg|\ \mathcal{F}_0\Bigg]\Bigg\}\\
&= e^{-rT}\Big[A_1^{UL} + A_2^{UL} + B_1^{UL} - B_2^{UL} + B_3^{UL} - B_4^{UL}\\
&\quad + C_1^{UL} - C_2^{UL} + C_3^{UL} - C_4^{UL}\\
&\quad + D_1^{UL} - D_2^{UL} - D_3^{UL} + D_4^{UL} + D_5^{UL} - D_6^{UL}\\
&\quad - D_7^{UL} + D_8^{UL}\Big],
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
A_1^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>D\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
A_2^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\le D\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
B_1^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^L\}}e^{-\int_0^T\lambda_t^u dt}\,\middle|\ \mathcal{F}_0\right],\\
B_2^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^L\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
B_3^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-L}V_T\mathbb{1}_{\{V_T\le De^L\}}e^{-\int_0^T\lambda_t^u dt}\,\middle|\ \mathcal{F}_0\right],\\
B_4^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-L}V_T\mathbb{1}_{\{V_T\le De^L\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
C_1^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^U\}}e^{-\int_0^T\lambda_t^l dt}\,\middle|\ \mathcal{F}_0\right],\\
C_2^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^U\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
C_3^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-U}V_T\mathbb{1}_{\{V_T\le De^U\}}e^{-\int_0^T\lambda_t^l dt}\,\middle|\ \mathcal{F}_0\right],\\
C_4^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-U}V_T\mathbb{1}_{\{V_T\le De^U\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
D_1^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^Ue^L\}}|\mathcal{F}_0\right],\\
D_2^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^Ue^L\}}e^{-\int_0^T\lambda_t^l dt}\,\middle|\ \mathcal{F}_0\right],\\
D_3^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^Ue^L\}}e^{-\int_0^T\lambda_t^u dt}\,\middle|\ \mathcal{F}_0\right],\\
D_4^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>De^Ue^L\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right],\\
D_5^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T\le De^Ue^L\}}|\mathcal{F}_0\right],\\
D_6^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T\le De^Ue^L\}}e^{-\int_0^T\lambda_t^l dt}\,\middle|\ \mathcal{F}_0\right],\\
D_7^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T\le De^Ue^L\}}e^{-\int_0^T\lambda_t^u dt}\,\middle|\ \mathcal{F}_0\right],\quad\text{and}\\
D_8^{UL} &= \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T\le De^Ue^L\}}e^{-\int_0^T(\lambda_t^u+\lambda_t^l)dt}\,\middle|\ \mathcal{F}_0\right].
\end{aligned}
$$

In addition, we also present the expression for the survival probability and the probability of default. Survival occurs when there is no default. Hence, for each case of final debt outcome, the value of the firm will be greater than the face value of debt:

$$\mathbb{1}_{\{V_T>D\}}\mathbb{1}_{\{\tau_U>T\}}\mathbb{1}_{\{\tau_L>T\}} + \mathbb{1}_{\{V_T>De^L\}}\mathbb{1}_{\{\tau_U>T\}}\mathbb{1}_{\{\tau_L\leq T\}} + \mathbb{1}_{\{V_T>De^U\}}\mathbb{1}_{\{\tau_U\leq T\}}\mathbb{1}_{\{\tau_L>T\}} + \mathbb{1}_{\{V_T>De^Ue^L\}}\mathbb{1}_{\{\tau_U\leq T\}}\mathbb{1}_{\{\tau_L\leq T\}}. \tag{21}$$

As usual, the survival probability is given by the expected value of the indicator functions, that is, the probabilities of the outcomes, thus yielding

$$\begin{aligned} SP_0^{UL}(V_0, D, T, \lambda_u, U, \lambda_l, L) &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{V_T>D\}}\mathbb{1}_{\{\tau_U>T\}}\mathbb{1}_{\{\tau_L>T\}}|\mathcal{G}_0\right] \\ &+ \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{V_T>De^L\}}\mathbb{1}_{\{\tau_U>T\}}\mathbb{1}_{\{\tau_L\leq T\}}|\mathcal{G}_0\right] \\ &+ \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{V_T>De^U\}}\mathbb{1}_{\{\tau_U\leq T\}}\mathbb{1}_{\{\tau_L>T\}}|\mathcal{G}_0\right] \\ &+ \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{V_T>De^Ue^L\}}\mathbb{1}_{\{\tau_U\leq T\}}\mathbb{1}_{\{\tau_L\leq T\}}|\mathcal{G}_0\right], \end{aligned} \tag{22}$$

and, consequently, the probability of default is calculated as

$$PD_0^{UL}(V_0, D, T, \lambda_u, U, \lambda_l, L) = 1 - SP_0^{UL}(V_0, D, T, \lambda_u, U, \lambda_l, L). \tag{23}$$

2.5 | Debt Increase and Subordinated Debt

The goal now is to study the possibility of a debt increase, but in the presence of a capital structure with subordinated debt in the spirit of Black and Cox (1976) and Gorton and Santomero (1990). This framework departs from the Merton (1974) base case and contemplates the presence of two kinds of debt: junior debt, with face value D^J, and senior debt, with face value D^S. In this study, we will consider only the possibility of increasing the senior debt and its impact on the junior debt.

The senior debt has priority in cases where the firm defaults, and this difference is highlighted in the payoffs across different scenarios. At time T, if the firm value, V_T, is greater than the sum of both debts—$V_T \geq D^S + D^J$—the two kinds of debt are paid in full. If $D^S + D^J > V_T \geq D^S$, the senior debt holder gets paid in full, while the junior debt holder collects $V_T - D^S$, that is, the senior debt holder has the priority to be paid in full, while the junior debt holder receives the remainder of the firm value. When $D^S > V_T$, the senior debt holder receives V_T while the junior debt holder receives zero, because in the case of default the senior debt holder has the priority to receive as much as possible, although not enough to be fully reimbursed, while the junior debt holder finds no value to compensate for his loan. As for the equity holders, they receive $V_T - D^S - D^J$ when the value is positive, and zero otherwise.

The above cases are summarized in Table 1.

Using the payoffs shown in Table 1 and expanding the model presented in Gorton and Santomero (1990) for the case of deadweight default losses, it is straightforward to show that the time-0 value for the junior debt holder is given by

$$\begin{aligned} &JB_0(V_0, D^J, D^S, T) \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[D^J\mathbb{1}_{\{V_T\geq D^S+D^J\}} + (\phi_{\text{dwl}}V_T - D^S)\mathbb{1}_{\{D^S+D^J>V_T\geq D^S\}}\middle| \mathcal{F}_0\right] \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[(\phi_{\text{dwl}}V_T - D^S)\mathbb{1}_{\{V_T\geq D^S\}} \\ &\quad - (\phi_{\text{dwl}}V_T - D^S - D^J)\mathbb{1}_{\{V_T\geq D^S+D^J\}}\Big| \mathcal{F}_0\Big] \\ &= E_0(V_0, D^S, T) - E_0(V_0, D^S + D^J, T) \end{aligned} \tag{24}$$

$$= B_0(V_0, D^S + D^J, T) - B_0(V_0, D^S, T), \tag{25}$$

that is, it can be interpreted as the difference between a call with strike D^S and a call with strike $D^S + D^J$, with the calls being computed via Equation (5) for a given deadweight default loss parameter ϕ_{dwl}, or understood as the difference between two risky debt claims evaluated through Equation (3) with face values equal to $D^S + D^J$ and D^S, respectively.

Let us now consider the possibility of an increase in the senior debt and analyze its impact on the junior debt. More specifically, we are admitting the possibility that, between times 0 and T, an increase in the senior debt (only) might occur through a process $d\lambda_t^u$ as defined in Equation (6), which has the solution shown in Equation (7). Again, to set a positive relation between the debt increase and the firm value, the correlation between the drifts of the firm value and the debt increases must be positive, that is, $\rho_{uV} > 0$. When this event occurs, the senior debt amount to be reimbursed will be D^Se^U, where, as usual, $U > 0$ corresponds to a debt increase. The remaining assumptions on the debt also hold in this case. As expected, the time-0 value of the junior debt must now be conditioned on the possibility of the event τ_U, yielding

$$\begin{aligned} &JB_0^U(V, D^J, D^S, T, \lambda_u, U) \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[\Big(\phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\geq D^S\}} - D^S\mathbb{1}_{\{V_T\geq D^S\}} \\ &\quad - \phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\geq D^S+D^J\}} + (D^S + D^J)\mathbb{1}_{\{V_T\geq D^S+D^J\}}\Big)\mathbb{1}_{\{\tau_U>T\}} \\ &\quad + \Big(\phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\geq D^Se^U\}} - D^Se^U\mathbb{1}_{\{V_T\geq D^Se^U\}} \\ &\quad - \phi_{\text{dwl}}V_T\mathbb{1}_{\{V_T\geq D^Se^U+D^J\}} + (D^Se^U + D^J)\mathbb{1}_{\{V_T\geq D^Se^U+D^J\}}\Big)\mathbb{1}_{\{\tau_U\leq T\}}\Big|\mathcal{G}_0\Big] \\ &= e^{-rT}\Big[JA_1^U - JA_2^U - JA_3^U + JA_4^U \\ &\quad + JB_1^U - JB_2^U - JB_3^U + JB_4^U - JB_5^U + JB_6^U + JB_7^U - JB_8^U\Big], \end{aligned} \tag{26}$$

TABLE 1 | Subordinated debt realized asset values at maturity.

	$V_T \geq D^S + D^J$	$D^S + D^J > V_T \geq D^S$	$D^S > V_T$
B_T^S (senior debt)	D^S	D^S	V_T
B_T^J (junior debt)	D^J	$V_T - D^S$	0
E_T (equity)	$V_T - D^S - D^J$	0	0

where

$$JA_1^U = \mathbb{E}_{\mathbb{Q}}\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \geq D^{\rm S}\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right],$$

$$JA_2^U = \mathbb{E}_{\mathbb{Q}}\left[D^{\rm S} \mathbb{1}_{\{V_T \geq D^{\rm S}\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right],$$

$$JA_3^U = \mathbb{E}_{\mathbb{Q}}\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \geq D^{\rm S}+D^{\rm J}\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right],$$

$$JA_4^U = \mathbb{E}_{\mathbb{Q}}\left[(D^{\rm S} + D^{\rm J}) \mathbb{1}_{\{V_T \geq D^{\rm S}+D^{\rm J}\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right],$$

$$JB_1^U = \mathbb{E}_{\mathbb{Q}}\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U\}} | \mathcal{F}_0\right],$$

$$JB_2^U = \mathbb{E}_{\mathbb{Q}}\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right],$$

$$JB_3^U = \mathbb{E}_{\mathbb{Q}}\left[D^{\rm S}{\rm e}^U \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U\}} | \mathcal{F}_0\right],$$

$$JB_4^U = \mathbb{E}_{\mathbb{Q}}\left[D^{\rm S}{\rm e}^U \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U\}} {\rm e}^{-\int_0^T \lambda_t^u ds}\right) \middle| \mathcal{F}\right],$$

$$JB_5^U = \mathbb{E}_{\mathbb{Q}}\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U + D^{\rm J}\}} | \mathcal{F}_0\right],$$

$$JB_6^U = \mathbb{E}_{\mathbb{Q}}\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U + D^{\rm J}\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right],$$

$$JB_7^U = \mathbb{E}_{\mathbb{Q}}\left[(D^{\rm S}{\rm e}^U + D^{\rm J}) \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U + D^{\rm J}\}} | \mathcal{F}_0\right], \quad \text{and}$$

$$JB_8^U = \mathbb{E}_{\mathbb{Q}}\left[(D^{\rm S}{\rm e}^U + D^J) \mathbb{1}_{\{V_T \geq D^{\rm S}{\rm e}^U + D^{\rm J}\}} {\rm e}^{-\int_0^T \lambda_t^u ds} \middle| \mathcal{F}_0\right].$$

3 | Solutions to the Cases

In this section, the solutions to the debt discount cases are solved. The following two propositions will be instrumental for it.

Proposition 1. *Under the financial model presented in Equation (1), coupled with Equations (6) and (14), and assuming that $\tau_U > 0$ and $\tau_L > 0$, the following expected value has the solution*

$$\begin{aligned}\mathbb{E}&\left[\phi_{\rm dwl} V_T \mathbb{1}_{\{V_T \leq D{\rm e}^U{\rm e}^L\}} {\rm e}^{-\int_0^T \left(\lambda_t^u \alpha_u + \lambda_t^l \alpha_l\right) dt} \middle| \mathcal{F}_0\right] \\ &= \phi_{\rm dwl} V_0 {\rm e}^{(r-q)T} H(\alpha_u, \alpha_l) F(\alpha_u, \alpha_l) \\ &\quad (1 - G(\alpha_u, \alpha_l, a_1(X))),\end{aligned} \tag{27}$$

where

$$G(\alpha_u, \alpha_l, a_1(X)) = N\left(\frac{a_1(X) - \alpha_u \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u} \int_0^T n_u(s,T)\, ds - \alpha_l \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l} \int_0^T n_l(s,T)\, ds}{\sigma_V \sqrt{T}}\right), \tag{28}$$

$$a_1(X) = \log(V_0/X) + \left(r - q + \frac{1}{2}\sigma_V^2\right)T, \tag{29}$$

$$X = D{\rm e}^U{\rm e}^L, \tag{30}$$

$$\begin{aligned}F(\alpha_u, \alpha_l) = \exp\Bigg[&\alpha_u\left(-\frac{\lambda_0^u}{\kappa_u} n_u(0,T) - \theta_u \int_0^T n_u(s,T)\, ds + \frac{\sigma_u^2}{2\kappa_u^2}\int_0^T n_u^2(s,T)\, du\right) \\ &+ \alpha_l\left(-\frac{\lambda_0^l}{\kappa_l} n_l(0,T) - \theta_l \int_0^T n_l(s,T)\, ds + \frac{\sigma_l^2}{2\kappa_l^2}\int_0^T n_l^2(s,T)\, du\right) \\ &+ \alpha_u \alpha_l \rho_{ul} \frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_0^T n_u(s,T) n_l(s,T)\, ds\Bigg],\end{aligned} \tag{31}$$

$$H(\alpha_u, \alpha_l) = \exp\left[-\alpha_u \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\, ds - \alpha_l \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\, ds\right], \tag{32}$$

and

$$n_x(t,T) = 1 - {\rm e}^{\kappa_x T}. \tag{33}$$

Proof. Please see Appendix A. ☐

Proposition 2. *Under the financial model presented by Equation (1), coupled with Equations (6) and (14), and assuming that $\tau_U > 0$ and $\tau_l > 0$, the following expected value has the solution*

$$\begin{aligned}\mathbb{E}&\left[D \mathbb{1}_{\{V_T > D{\rm e}^U{\rm e}^L\}} {\rm e}^{-\int_0^T \left(\lambda_t^u \alpha_u + \lambda_t^l \alpha_l\right) dt} \middle| \mathcal{F}_0\right] \\ &= DF(\alpha_u, \alpha_l) G(\alpha_u, \alpha_l, a_2(X)),\end{aligned} \tag{34}$$

with

$$a_2(X) = \log(V_0/X) + \left(r - q - \frac{1}{2}\sigma_V^2\right)T, \tag{35}$$

$$X = D{\rm e}^U{\rm e}^L, \tag{36}$$

and where the functions $G(\alpha_u, \alpha_l, x)$ and $F(\alpha_u, \alpha_l)$ are defined as in Proposition 1.

Proof. Please see Appendix B. ☐

The formula for the possibility of debt increase and/or decrease is the first one to be solved. The remaining formulae follow similar steps.

Proposition 3 (Debt increase and/or decrease). *Under the financial model shown in Equation (1), with the debt changes being ruled by Equations (6) and (14), and assuming that $\tau_U > 0$*

and $\tau_L > 0$, then the value of a bond at time-0 on a firm under the possibility of suffering a debt increase and/or decrease is equal to Equation (20), where

$$
\begin{aligned}
A_1^{UL} &= DF(1,1)G(1,1,a_2(D)),\\
A_2^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0F(1,1)H(1,1)(1-G(1,1,a_1(D\mathrm{e}^U\mathrm{e}^J)),\\
B_1^{UL} &= DF(1,0)G(1,0,a_2(D\mathrm{e}^L)),\\
B_2^{UL} &= DF(1,1)G(1,1,a_2(D\mathrm{e}^L)),\\
B_3^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0F(1,0)H(1,0)(1-G(1,0,a_1(D\mathrm{e}^L))),\\
B_4^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0F(1,1)H(1,1)(1-G(1,1,a_1(D\mathrm{e}^L))),\\
C_1^{UL} &= DF(0,1)G(0,1,a_2(D\mathrm{e}^U)),\\
C_2^{UL} &= DF(1,1)G(1,1,a_2(D\mathrm{e}^U)),\\
C_3^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-U}V_0\mathrm{e}^{-(r-q)T}F(0,1)H(0,1)(1-G(0,1,a_1(D\mathrm{e}^U))),\\
C_4^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-U}V_0\mathrm{e}^{-(r-q)T}F(1,1)H(1,1)(1-G(1,1,a_1(D\mathrm{e}^U))),\\
D_1^{UL} &= DG(0,0,a_2(D\mathrm{e}^U\mathrm{e}^J)),\\
D_2^{UL} &= DF(0,1)G(0,1,a_2(D\mathrm{e}^U\mathrm{e}^J)),\\
D_3^{UL} &= DF(1,0)G(1,0,a_2(D\mathrm{e}^U\mathrm{e}^J)),\\
D_4^{UL} &= DF(1,1)G(1,1,a_2(D\mathrm{e}^U\mathrm{e}^J)),\\
D_5^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-U}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0(1-G(0,0,a_1(D\mathrm{e}^U\mathrm{e}^J))),\\
D_6^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-U}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0F(0,1)H(0,1)\\
&\quad (1-G(0,1,a_1(D\mathrm{e}^U\mathrm{e}^J))),\\
D_7^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-U}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0F(1,0)H(1,0)\\
&\quad (1-G(1,0,a_1(D\mathrm{e}^U\mathrm{e}^J))),\\
D_8^{UL} &= \phi_{\text{dwl}}\mathrm{e}^{-U}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0F(1,1)H(1,1)\\
&\quad (1-G(1,1,a_1(D\mathrm{e}^U\mathrm{e}^J))),
\end{aligned}
$$

whereas the functions $F(\alpha_u,\alpha_l)$, $G(\alpha_u,\alpha_l,x)$, and $H(\alpha_u,\alpha_l)$ are defined in Proposition 1.

Proof. For A_1^{UL}, it can be observed that the expected value matches Proposition 2 if $\alpha_u=1, \alpha_l=1, U=0$, and $L=0$. Therefore, the solution is

$$A_1^{UL} = DF(1,1)G(1,1,d_2(D)).$$

In the case of A_2^{UL}, the expected value can be matched to Proposition 1 if $\alpha_u=1$ and $\alpha_l=1$. Hence, the solution is

$$A_2^{UL} = \phi_{\text{dwl}}\mathrm{e}^{(r-q)T}V_0H(1,1)F(1,1)(1-G(1,1,a_1(D_0\mathrm{e}^U\mathrm{e}^L))).$$

As for B_1^{UL}, the expected value is matched by Proposition 2 if $\alpha_u=1, \alpha_l=0$, and $U=0$. Therefore, the solution is

$$B_1^{UL} = DF(1,0)G(1,0,a_2(D_0\mathrm{e}^L)).$$

The remainder of the elements follow the same rationale. □

Proposition 4 (Debt increase). *Under the financial model presented by Equation (1), with the debt changes being ruled by Equation (6) and assuming that $\tau_U > 0$, the value of a bond at time-0 on a firm under the possibility of suffering a debt increase is equal to Equation (11), where*

$$
\begin{aligned}
A_1^U &= DF(1,0)G(1,0,a_2(D)),\\
A_2^U &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0F(1,0)H(1,0)(1-G(1,0,a_1(D))),\\
B_1^U &= DG(0,0,a_2(D\mathrm{e}^U)),\\
B_2^U &= DF(1,0)G(1,0,a_2(D\mathrm{e}^U),\\
B_3^U &= \phi_{\text{dwl}}\mathrm{e}^{-U}\mathrm{e}^{-(r-q)T}V_0(1-G(0,0,a_1(D\mathrm{e}^U))),\\
B_4^U &= \phi_{\text{dwl}}\mathrm{e}^{-U}\mathrm{e}^{-(r-q)T}F(1,0)H(1,0)V_0(1-G(1,0,a_1(D\mathrm{e}^U))),
\end{aligned}
$$

whereas the functions $F(\alpha_u,\alpha_l)$, $G(\alpha_u,\alpha_l,x)$, and $H(\alpha_u,\alpha_l)$ are defined in Proposition 1.

Proof. This proof is similar to the one used in Proposition 3 and is, therefore, omitted. □

Proposition 5 (Debt decrease). *Under the financial model presented by Equation (1), with the debt changes being ruled by Equation (6) and assuming that $\tau_L > 0$, the value of a bond at time-0 on a firm under the possibility of suffering a debt decrease is equal to Equation (19), where*

$$
\begin{aligned}
A_1^L &= DF(0,1)G(0,1,a_2(D)),\\
A_2^L &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0F(0,1)H(0,1)(1-G(0,1,a_1(D))),\\
B_1^L &= DG(0,0,a_2(D\mathrm{e}^L)),\\
B_2^L &= DF(0,1)G(0,1,a_2(D\mathrm{e}^L),\\
B_3^L &= \phi_{\text{dwl}}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}V_0(1-G(0,0,a_1(D\mathrm{e}^L))),\\
B_4^U &= \phi_{\text{dwl}}\mathrm{e}^{-L}\mathrm{e}^{-(r-q)T}F(0,1)H(0,1)V_0(1-G(0,1,a_1(D\mathrm{e}^L))),
\end{aligned}
$$

whereas the functions $F(\alpha_u,\alpha_l)$, $G(\alpha_u,\alpha_l,x)$, and $H(\alpha_u,\alpha_l)$ are defined in Proposition 1.

Proof. This proof is similar to the one used in Proposition 3 and is, therefore, omitted. □

Proposition 6 (Junior debt under senior debt increase). *Under the financial model presented by Equation (1), with the debt changes being ruled by Equation (14) and assuming that $\tau_L > 0$, the value at time-0 of a junior bond on a firm under the possibility of suffering a debt decrease on the senior debt is equal to Equation (26), where*

$$
\begin{aligned}
JA_1^U &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0F(1,0)H(1,0)G(1,0,a_1(D^S)),\\
JA_2^U &= D^SF(1,0)G(1,0,a_2(D^S)),\\
JA_3^U &= \phi_{\text{dwl}}V_0\mathrm{e}^{-(r-q)T}F(1,0)H(1,0)G(1,0,a_1(D^S+D^J)),\\
JA_4^U &= (D^S+D^J)F(1,0)G(1,0,a_2(D^S+D^J)),\\
JB_1^U &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0G(0,0,a_1(D^S\mathrm{e}^U)),\\
JB_2^U &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0F(1,0)H(1,0)G(1,0,a_1(D^S\mathrm{e}^U)),\\
JB_3^U &= D^S\mathrm{e}^UG(0,0,a_2(D^S\mathrm{e}^U)),\\
JB_4^U &= D^S\mathrm{e}^UF(1,0)G(1,0,a_2(D^S\mathrm{e}^U)),\\
JB_5^U &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0G(0,0,a_1(D^S\mathrm{e}^U+D^J)),\\
JB_6^U &= \phi_{\text{dwl}}\mathrm{e}^{-(r-q)T}V_0F(1,0)H(1,0)G(1,0,a_1(D^S\mathrm{e}^U+D^J)),\\
JB_7^U &= (D^S\mathrm{e}^U+D^J)G(0,0,a_2(D^S\mathrm{e}^U+D^J)),\\
JB_8^U &= (D^S\mathrm{e}^U+D^J)F(1,0)G(1,0,a_2(D^S\mathrm{e}^U+D^J)),
\end{aligned}
$$

whereas the functions $F(\alpha_u, \alpha_l)$, $G(\alpha_u, \alpha_l, x)$, and $H(\alpha_u, \alpha_l)$ are defined in Proposition 1.

Proof. This proof is similar to the one used in Proposition 3 and is, therefore, omitted. □

4 | Result Analysis

In the analyses that follow, the firm value is normalized to $V_0 = 1$, while the two main explored debt levels are $D = \{0.75, 0.50\}$, the high-leverage and low-leverage cases. The firm value volatility is assumed to be $\sigma_V = 20\%$, the risk-free rate is $r = 2\%$, the total payout on debt and equity holders, q is set at zero, and the studied levels of the recovery values are $\phi_{\text{dwl}} = \{1, 0.70\}$, a full and partial recovery. The debt increase is $U = \log(1.30)$, which corresponds to an increase of 30% over the initial amount of debt and the debt decrease is $L = \log(1/1.30)$, which corresponds to a decrease of approximately 23.1%. As for the increase and decrease events, we set the volatility of both processes to $\sigma_u = \sigma_l = 0.30$. The initial and mean reverting value of the processes is the same in both cases $\lambda_0^u = \lambda_0^l = \theta_u = \theta_l = 0.20$. Thus, both the initial and the long-term average imply that in a small time interval Δ, the possibility of either an increase or a decrease occurring on the debt is approximately 0.20Δ. The speed of reversion of both processes is $\kappa_u = \kappa_l = 3$. Perfect correlations are assumed between the firm value and the debt increase and debt decrease processes, positive in the debt increase $\rho_{uV} = 1$, and negative in the debt decrease $\rho_{lV} = -1$. As for the correlation between both processes, it is assumed to be zero, $\rho_{ul} = 0$. In the subordinated debt case, the total debt is equally distributed among the junior debt and the senior debt, such that $D^{\text{J}} = D^{\text{S}} = 0.375$ in the high-leverage case and $D^{\text{S}} = D^{\text{J}} = 0.25$ in the low-leverage case.

We start with an analysis of the overall model, exploring different maturities. After that, we present a comparative statics analysis of the different variables introduced. Lastly, the particular case of debt seniority is presented.

4.1 | Numerical Analysis

First, we study the direct impact of the proposed dynamic debt setting on the debt values and corresponding credit spreads. In Table 2, the various values of the bonds and the corresponding spreads are presented, considering $T = 15$. The spreads in all these cases are computed in the usual manner with $\mathcal{S} = -\frac{1}{T}\log(\frac{BV_0}{De^{-rT}})$, where BV_0 stands for bond value, which will be calculated for the different cases.

In the base case, focusing in the case of $D/V = 0.75$ and $\phi = 1$, B_0, it is possible to observe that without the possibility of debt changes, the bond values are smaller and the spreads are higher for greater amounts of initial debt and lower recovery values. This pattern holds when the debt can change.

Comparing with the base case, the possibility of increasing debt, B_0^U, translates into a spread increment of 50 basis points to 142 and the possibility of decreasing debt, B_0^L, into a decline of 37 basis points to 55. These results are expected, given increases in debt lead to an increased probability of default, and when those defaults occur, the amount to be received from the original debt must be shared with the new debt holder(s); the reverse applies to debt decreases.

As for how the debt value reacts in the presence of the possibility of both the increase and the decrease in debt, the change is close to null, with the spread decreasing by one point. Given the studied time period is 15 years and the chosen parameters for the amount of debt to be increased and decreased, by then, this is expected, as with high likelihood, the increases and decreases in the debt have returned it to its original value. In the subordinated debt case, although the bond values have a difficult comparison given the face value of debt being different, the high spreads highlight the increased risk of the junior debt in relation to the base case.

Regarding how the parameters affect the credit spreads over time, Figures 1 and 2 show how these behave in the low-leverage case—$D = 0.50$—and in the high-leverage case—$D = 0.75$—respectively.

By observing the figures, the results from Table 2 are confirmed, thus validating that the proposed setup is able to emulate the empirical behavior predicted in Flannery et al. (2012) and Maglione (2022), where leverage change expectations (here modeled through intensity processes) are crucial for explaining credit spreads. For all maturities, it can be seen that expectations of future increases in leverage are linked to augmented credit spreads, while expectations of future decreases reduce spreads.

TABLE 2 | Bond values and spreads for the studied cases at $T = 15$.

	$D/V = 0.75$				$D/V = 0.50$			
	$G_{\phi=1}$		$G_{\phi=0.70}$		$G_{\phi=1}$		$G_{\phi=0.70}$	
	Bond value	**Spread**	**Bond value**	**Spread**	**Bond value**	**Spread**	**Bond value**	**Spread**
B_0	0.4842	92	0.4464	146	0.3492	39	0.3350	67
B_0^U	0.4489	142	0.4009	218	0.3350	67	0.3143	110
B_0^L	0.5114	55	0.4841	92	0.3589	21	0.3500	38
B_0^{UL}	0.4848	91	0.4470	145	0.3496	39	0.3355	66
JB_0^L	0.1947	237	0.1580	376	0.1576	108	0.1412	181

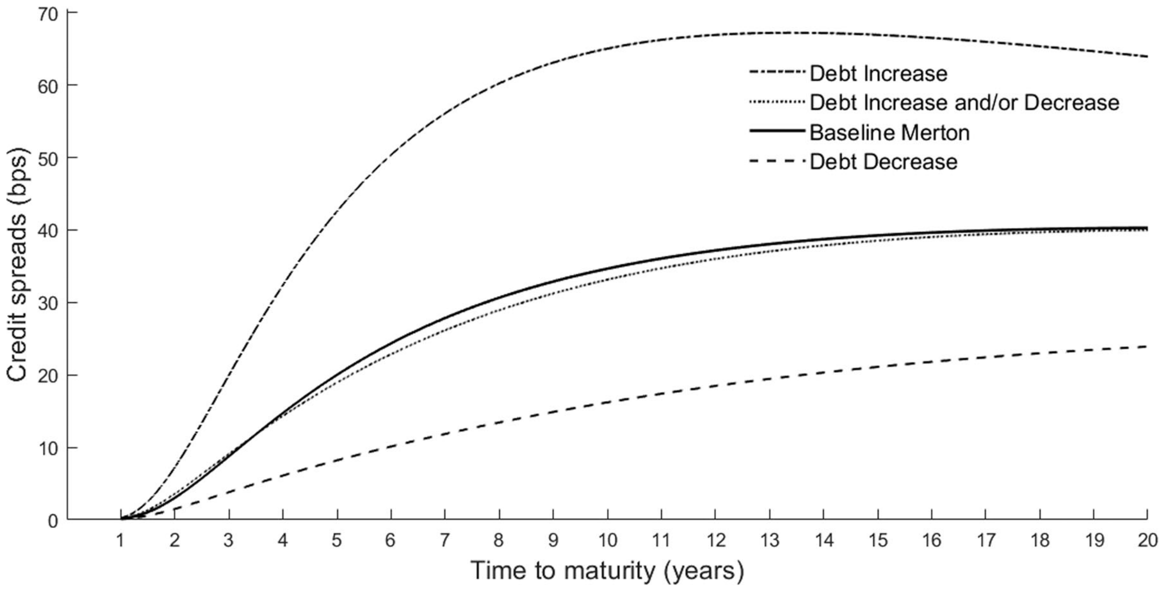

FIGURE 1 | Credit spreads of the baseline Merton (1974) model, the possibility to increase debt, the possibility to decrease debt, and the possibility to increase and/or decrease debt with the standard parameters for the low-leverage case where $D = 0.50$.

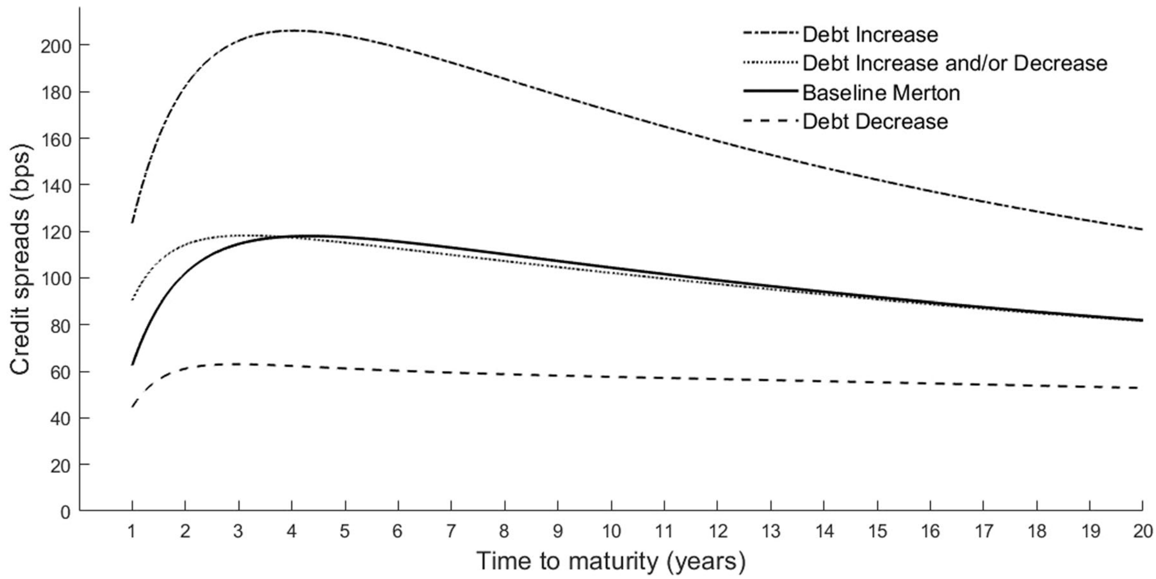

FIGURE 2 | Credit spreads of the baseline Merton (1974) model, the possibility to increase debt, the possibility to decrease debt, and the possibility to increase and/or decrease debt with the standard parameters for the high-leverage case where $D = 0.75$.

As for the shape of the curves, the possibility of debt increases leads to a greater magnitude of the hump-shaped curves that come from the Merton (1974) base case, while in the case where there is the possibility of a debt decrease, the curves approach a flat shape.

As for the comparison between the baseline Merton (1974) case and the increase and/or decrease possibilities for the debt case, we note that over the short term, the uncertainty over the debt level leads to increased spreads; over the medium term, the spreads are lower, with the impact of the possibility of the debt decrease being greater than the one from the possibility of the debt increase; over the long term, the spreads tend to converge, as the possibility of both the increase and the decrease having occurred is very high and the debt has returned to the initial amount with high likelihood.

Still, further conclusions can be reached by exploring the model parameters. In Figure 3, we recalibrate the terms in relation to Figure 2, making the debt decrease more likely by changing the initial and long-term average of the corresponding process to $\theta_l = \lambda_0^l = 0.30$ and the debt increase less likely by selecting those debt increase process parameters to $\theta_u = \lambda_0^u = 0.10$. With this, for the higher leverage case ($D = 0.75$), and for the model which combines increases and decreases in leverage, we are able to replicate the conclusions of Gemmill (2002), which point to reductions in leverage as key for the Merton model to obtain an upward slopping spread curve, as in real-world data. Also note that using the debt reduction in isolation produces a

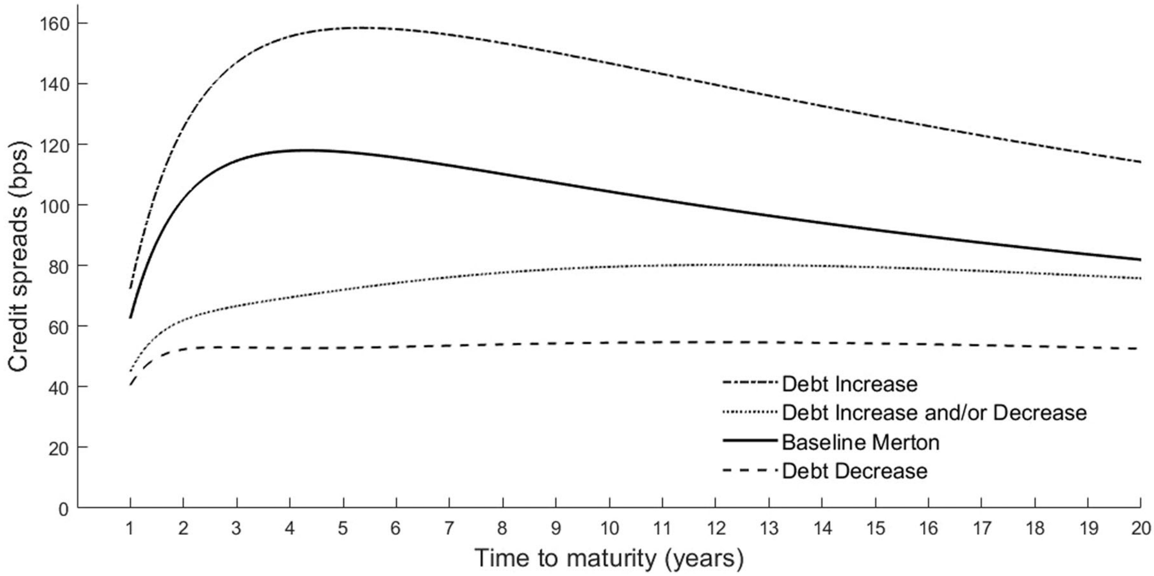

FIGURE 3 | Credit spreads of the baseline Merton (1974) model and the possibility to increase and/or decrease debt with the standard parameters for the high-leverage case, where $D = 0.75$ with upward slopping curve parameterization.

mostly flat curve, while combining that with the debt increases results in a curve approximating the intended shape.

4.2 | Comparative Statics

Now, we analyze the individual variables added by the dynamic debt model and their impact on the credit spreads. Besides using a maturity date of $T = 5$, to capture the effects at the medium term, the leverage is selected in the lower level case, $D = 0.50$, while the remainder of the parameters, unless stated otherwise, follow the baseline. We focus on the case of the bond with the possibility of increasing and decreasing, as in Equation (20), B_0^{UL}.

Starting with Figure 4, it is clear the amounts by how much the debt is increased and decreased, U and L, impact the spreads. In the case of the increase in debt, the greater the amount, the greater the spreads. As for the amount of the debt reduction, which can be observed as we move right in the top abscissa axis, when the write-down occurs, the spreads suffer a greater reduction the more intense it is. Overall, upward changes in the debt level increase spreads and downward changes reduce spreads.

In Figure 5, we approach the long-term averages of the hazard processes, θ_u and θ_l. The initial values for the processes, λ_0^u and λ_0^l, are set to match the corresponding long-term average in each case. In the instance of the debt increase, represented by θ_u, the greater the parameter is, the greater are the spreads. In other words, given that θ_u (and λ_0^u) is higher, the greater is the probability of triggering a debt increase, which in turn increases debt, reducing the bond value and increasing the credit spreads. In the case of θ_l, the same effect works in the opposite direction. As the average value of the debt decrease process (and its initial value) is increased, the chances of a debt decrease are greater, thus contributing to a reduction in the credit spreads.

In Figure 6, we take a first approach to the value of correlations between the hazard processes and the process of firm value (V_t), ρ_{uV} and ρ_{lV}. As can be seen, increasing the intensity of each correlation decreases credit spreads. In the case of the debt increase and $\rho_{uV}(> 0)$, the debt increase becomes more likely as the firm value increases and less likely when it decreases. Thus, when the firm is farther from default—and debt increases are less likely to induce a credit event—the debt is more likely to increase; when it is closer to default—and debt increases are more likely to provoke financial distress—it is less likely to do so.

The situation is reversed for the debt decrease, as it becomes more likely as the firm decreases in value, thus occurring when it is most important to stave of default, thus decreasing credit spreads. As shown below, the study of the correlations is also relevant when studying the variance and the speed of the reversion.

As for the volatilities of each hazard process, σ_u and σ_l, the effects are studied in Figure 7, doing so with different values for the correlations of each process in relation to the process of the firm value. We first inspect the case where the correlations are null, $\rho_{lV} = \rho_{uV} = 0$, presented in the panel on the left-hand side. Here, increasing the variance has the reverse effect on the average values of the intensity process. For instance, by increasing the volatility of the debt increase process (λ_u), the spreads are decreased. This can be explained by the nature of the Vasicek process, as presented in Equation (6), while recalling that $P(\tau_u > T) = \mathbb{E}\left[e^{-\int_0^T \lambda_t^u \, dt}\right]$. Given the nature of the exponential function, increasing the variance of the hazard process λ_l tends to have an asymmetrical effect—great increases in the value tend to approach values close to zero in the exponential function, while great decreases are not subject to such a bound. This asymmetry results in a greater variance of the process bringing about in a lower expected value for the probability of the debt increase to occur, $P(\tau_u \leq T) = 1 - P(\tau_u > T)$.

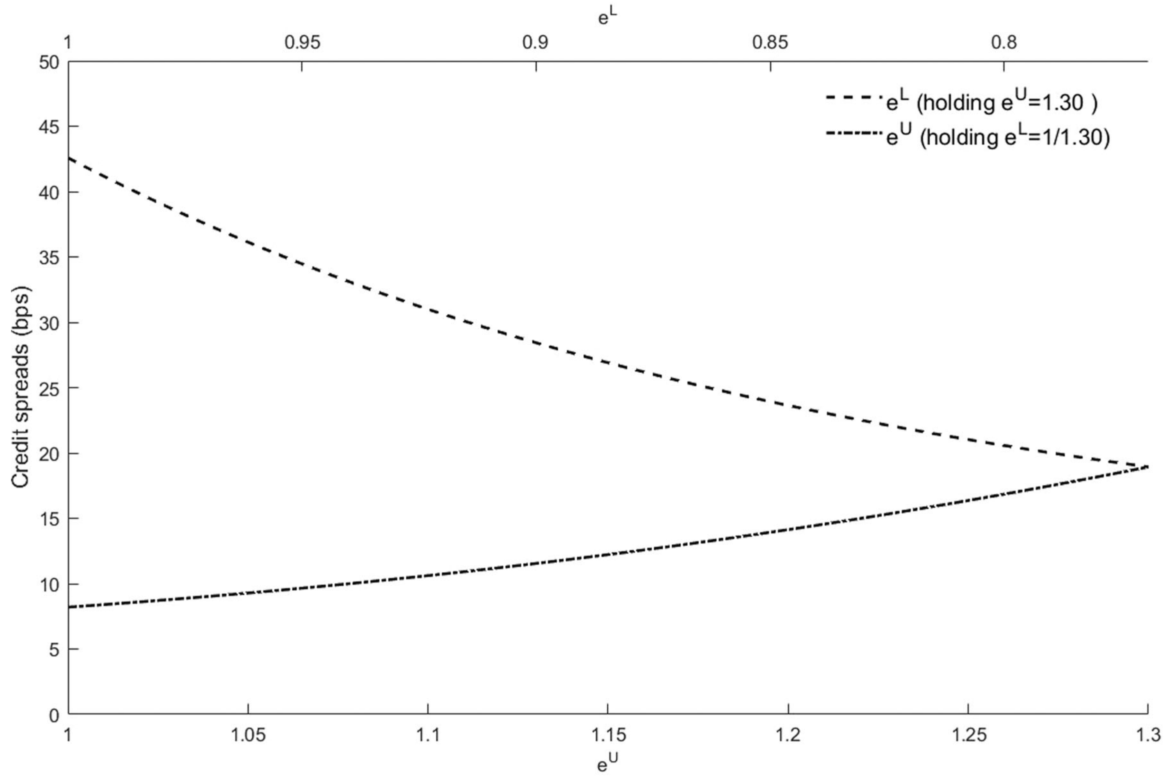

FIGURE 4 | Credit spreads of a bond with the possibility to increase and/or decrease debt with the standard parameters for the low-leverage case, $D = 0.50$. Variations over e^U (bottom abscissa) and e^L (top abscissa).

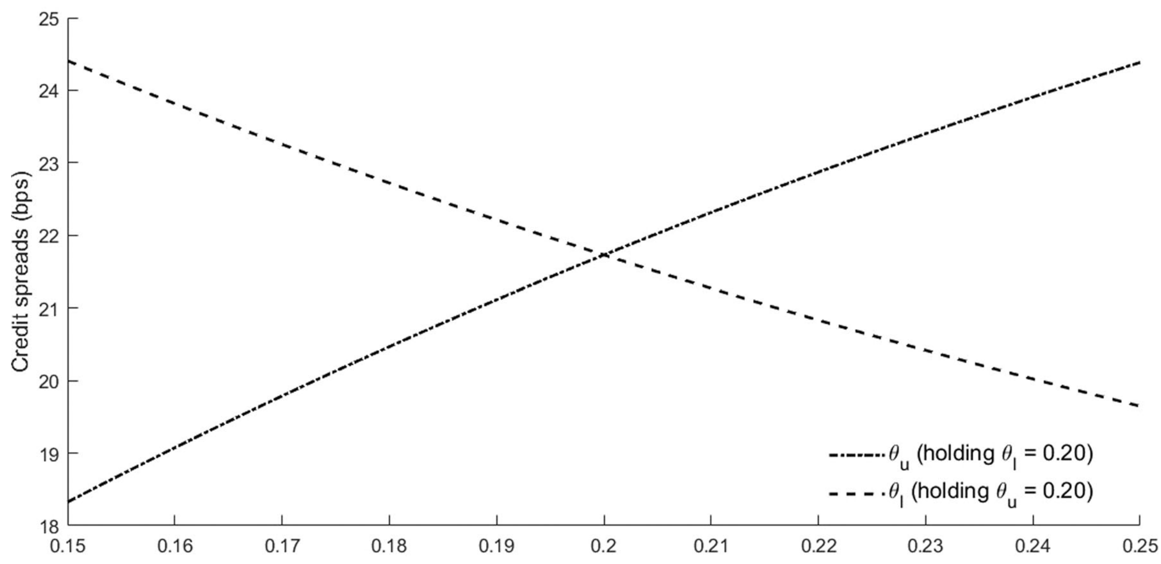

FIGURE 5 | Credit spreads of bond with the possibility to increase and/or decrease debt with the standard parameters for the low-leverage case, $D = 0.50$. Variations over $\theta_u = \lambda_0^u$ and $\theta_l = \lambda_0^l$.

This can also be noticed in Equation (A8), where increases in volatility have the reverse effect of the increases in the average of the hazard process. As for the debt decrease variance, it follows that increased variance of the hazard process tends to lead toward increases in spreads, as the effect presented above leads to a decrease in the probability of the debt being written down.

As we move to the panel on the right-hand side, in the instance of the debt increase, the effect of the increased volatility leads in the same direction as on the left-hand side. More significantly, the effect is reversed in the case of a decrease in debt. This reversal is explained by the negative correlation, which in this case is equal to -1. Looking at Equation (14), it can be seen that the variance, σ_l, is multiplied by $dW_{t,l}^{\mathbb{Q}}$, where the correlation link is established with the process of V_t, perfectly negative in this case. In other words, σ_l enhances the effect of the negative correlation, which, as studied in Figure 6, reduces the spreads resulting from the debt decrease. This relation, between σ_l and ρ_{lV}, is evident, for instance, in the expression of $H(\alpha_u, \alpha_l)$ from Proposition 1.

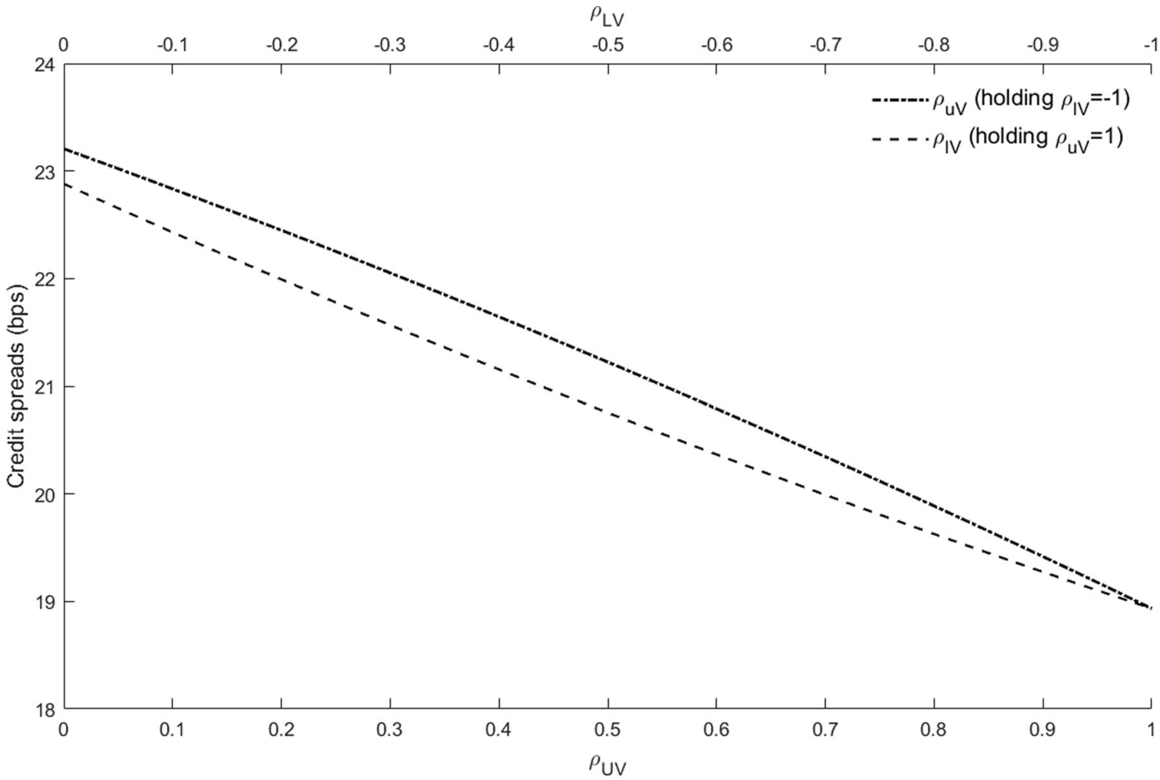

FIGURE 6 | Credit spreads of bond with the possibility to increase and/or decrease debt with the standard parameters for the low-leverage case, $D = 0.50$. Variations over ρ_{uV} (bottom abscissa) and ρ_{lV} (top abscissa).

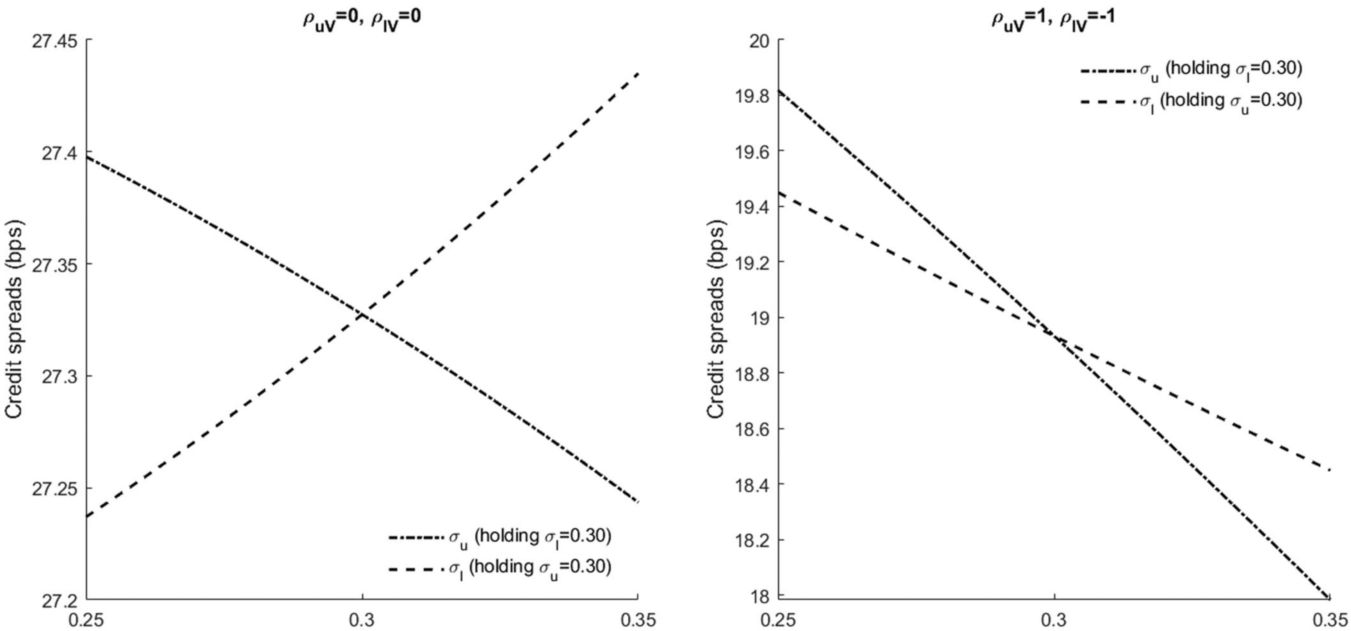

FIGURE 7 | Credit spreads of a bond with the possibility to increase and/or decrease debt with the standard parameters for the low-leverage case where $D = 0.50$. Variations over σ_u and σ_l, while considering on the left-hand side $\rho_{uV} = \rho_{lV} = 0$ and on the right-hand side $\rho_{uV} = 1$ and $\rho_{lV} = -1$.

As for the final variable, the speeds of reversion of the hazard processes, $\varkappa_u$ and $\varkappa_l$, are analyzed in Figure 8, once again splitting between two sets for the correlation values. As it can be seen on the left-hand side, where $\rho_{uV} = \rho_{lV} = 0$, increasing the reversion parameters increases the spreads when relating to debt ratchet and decreases the spreads with respect to the debt decrease. This can be interpreted in relation to the variance parameters—while σ_u and σ_l steer the values away from the average value of the process, high values for the reversion speed prevent great deviations away from the average value. In other words, increasing the speed of reversion to the mean counterweights the effect of the variance. Thus, the effect from Figure 7 is nullified, leading to increased probabilities of debt ratchets and write-downs, generating, respectively, credit spread

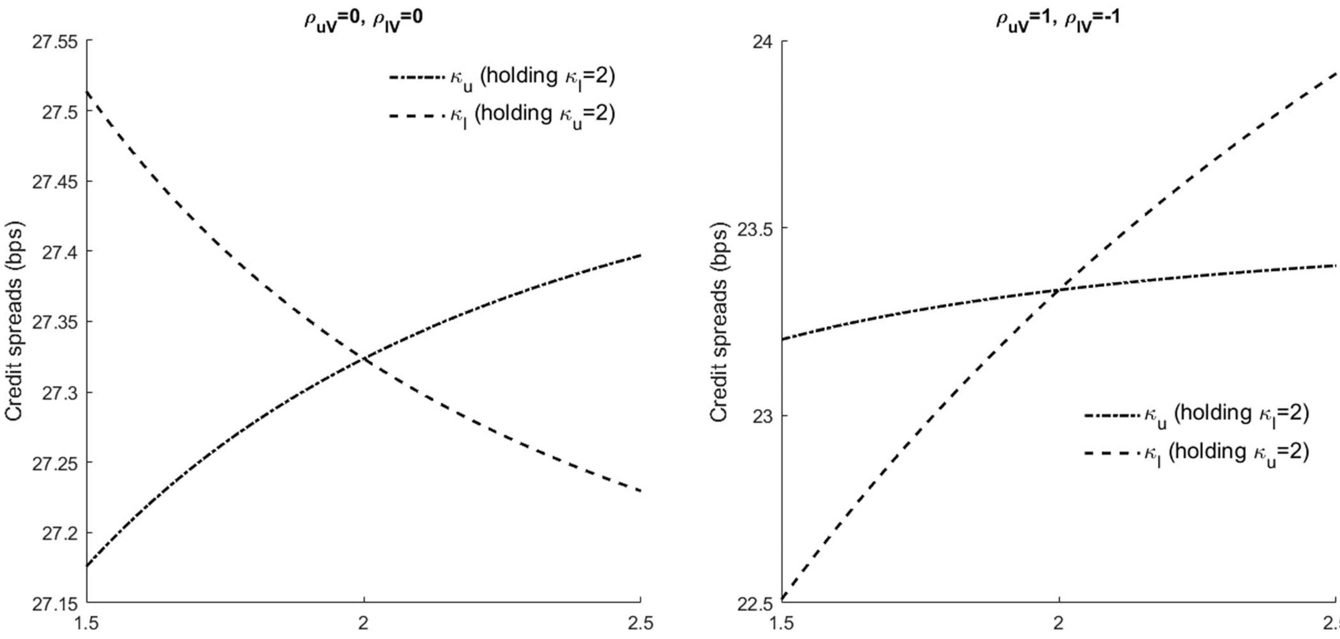

FIGURE 8 | Credit spreads of bond with the possibility to increase and/or decrease debt with the standard parameters for the low-leverage case, $D = 0.50$. Variations over κ_u and κ_l, while considering on the left-hand side $\rho_{uV} = \rho_{lV} = 0$ and on the right-hand side $\rho_{uV} = 1$ and $\rho_{lV} = -1$.

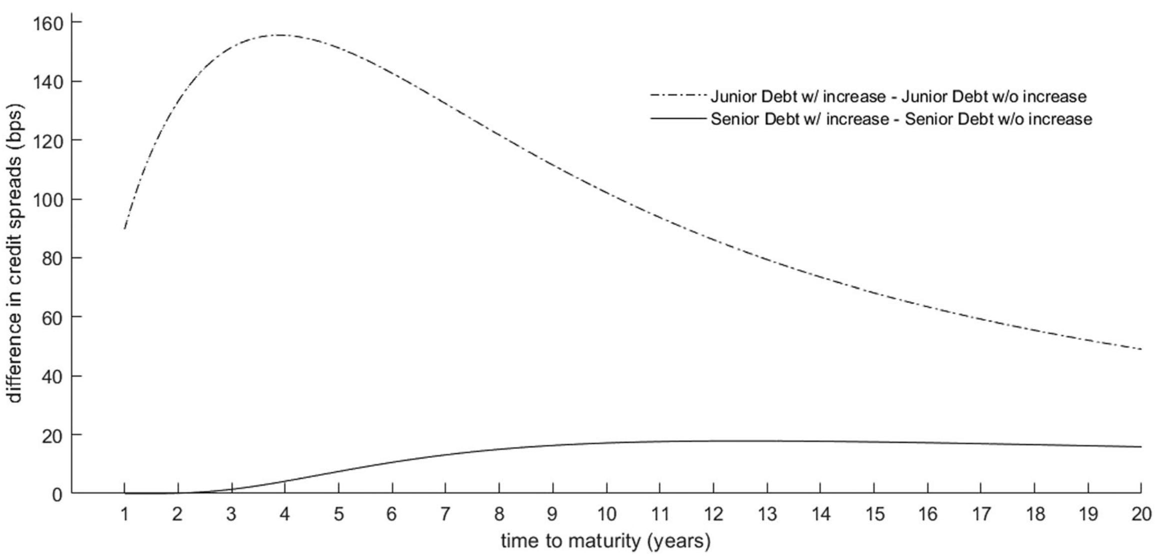

FIGURE 9 | Junior debt and senior debt spreads with the possibility of an increase minus, respectively, the Junior and Senior debt spreads without the possibility of an increase. $D^S = D^J = 0.375$.

increases and decreases. This pattern, of κ_u and κ_l serving as a counterweight to σ_u and σ_l, can be noticed in several equations from Proposition 1, by noting the reversion speed in the fractions' denominators.

Analyzing the case where $\rho_{uV} = 1$ and $\rho_{lV} = -1$, on the right-hand side, a similar pattern to that of the previous figure emerges. Notoriously, the negative correlation results in increases in the credit spreads—reversing the effect of the case with a null correlation. This can be attributed to the speed of reversion nullifying the effect of the negative correlation transmitted through $dW_{t,l}^{\mathbb{Q}}$ and multiplied by σ_l. That is, the spread-decreasing effect that comes from $\rho_{uV} = -1$ (and transmitted through σ_l) is quickly reversed by the higher κ_u. Given the containment of the spread decrease that would come from the negative correlation, the debt spreads are increased.

4.3 | Debt Seniority

Finally, we study the subordinated debt case under the possibility of a debt increase. Following what is presented in eq. (1) of Gorton and Santomero (1990), where the senior debt value is calculated as regular debt, the values for the senior debt are calculated as in the B^U case. The debt values are set to $D^S = D^J = 0.375$.

As it can be noted in Figure 9, as expected, the possibility of the debt ratchet increases both spreads. While the possibility of the increase is over the senior debt amount, the increase in the spreads in the junior debt is significantly larger than the one in the former debt. Therefore, while not being the debt that is with the possibility of an increase, the junior debt is the most affected. This results from its low seniority and thus lower priority to receive the remainder of the value of the firm in the case of default.

5 | Conclusions

In this paper, the concept of debt changing in discrete jumps of Das and Kim (2015) was explored in a different fashion. Instead of relying on the firm value crossing certain barriers, the changes in debt have their own process, although it can still be linked to the firm value through correlation of the processes.

Through changes of measure, we obtain the formulae for various cases that depart from the baseline Merton model: for the possibility of an increase, the possibility of a decrease, and a combination of both. We observe that the possibility of increasing magnifies the credit spreads, while the possibility of decreasing reduces them. The case where debt can increase and/or decrease can be ambiguous, with the uncertainty over the future debt value being able to increase or decrease spreads, even when the debt value is equal after the changes in debt. We also study the impact of the various model parameters on the credit spreads, finding, in particular, that the inclusion of correlations has a deep influence on the debt changes.

Finally, given the flexibility of this dynamic debt framework, we extend it to the subordinated debt case of Gorton and Santomero (1990), and observe the impact of increasing the senior over itself and the junior debt. We notice that the impact over the junior debt is greater, although the senior debt is the one which is the most affected by the possibility of the increase.

Author Contributions

João Miguel Reis and José Carlos Dias conceived the study and developed the methodology, conducted the analysis, wrote and revised.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

References to existing literature are included for contextual and theoretical support; however, no external data sources were utilized in this paper.

Endnotes

[1] For details on the solution, see, for instance, Musiela and Rutkowski (2005, Chapter 10).

[2] See, for instance, Protter (2003, Theorem IV.65).

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Appendix A

Proof of Proposition 1

Let us first define $n_u(0,T) := 1 - \mathrm{e}^{-\kappa_u T}$. Using Equation (7) and the insights of Musiela and Rutkowski (2005, Proposition 10.1.2), it follows that

$$\int_0^t \lambda_t^u \, dt = \lambda_0^u \int_0^t \mathrm{e}^{-\kappa_u t} dt + \theta_u \int_0^t (1 - \mathrm{e}^{-\kappa_u t}) \, dt + \sigma_u \int_0^T \int_0^t \mathrm{e}^{-\kappa_u (t-s)} dW_{s,u}^{\mathbb{Q}} dt. \tag{A1}$$

Notice that

$$\int_0^T \mathrm{e}^{-\kappa_u t} dt = \frac{1}{\kappa_u}(1 - \mathrm{e}^{-\kappa_u T}) = \frac{1}{\kappa_u} n_u(0,T) \tag{A2}$$

and

$$\int_0^t (1 - \mathrm{e}^{-\kappa_u t}) \, dt = \int_0^t dt - \int_0^t \mathrm{e}^{-\kappa_u t} dt = T - \frac{1}{\kappa_u} n_u(0,T). \tag{A3}$$

Using the stochastic Fubini theorem,[2]

$$\begin{aligned}\int_0^T \int_0^t \mathrm{e}^{-\kappa_u (t-s)} \, dW_{s,u}^{\mathbb{Q}} \, dt &= \int_0^T \int_0^t \mathrm{e}^{-\kappa_u t} \mathrm{e}^{\kappa_u s} dW_{s,u}^{\mathbb{Q}} \, dt \\ &= \int_0^T \mathrm{e}^{\kappa_u s} \int_s^T \mathrm{e}^{-\kappa_u t} dt W_{s,u}^{\mathbb{Q}} = \tfrac{1}{\kappa_u} \int_0^T (1 - \mathrm{e}^{-\kappa_u (T-s)}) \, dW_{s,u}^{\mathbb{Q}} \\ &= \tfrac{1}{\kappa_u} \int_0^T n(s,T) \, dW_{s,u}^{\mathbb{Q}}. \end{aligned} \tag{A4}$$

Combining Equations (A1)–(A4) yields

$$\int_0^t \lambda_t^u \, dt = \frac{\lambda_0^u}{\kappa_u} n_u(0,T) + \theta_u \int_0^t n_u(s,T) \, ds + \frac{\sigma_u}{\kappa_u} \int_0^T n_u(s,T) \, dW_{s,u}^{\mathbb{Q}}. \tag{A5}$$

Following the same rationale for Equation (15), with $n_l(0,T) := 1 - \mathrm{e}^{-\kappa_l T}$, we obtain

$$\int_0^t \lambda_t^l \, dt = \frac{\lambda_0^l}{\kappa_l} n_l(0,T) + \theta_l \int_0^t n_l(s,T) \, ds + \frac{\sigma_l}{\kappa_l} \int_0^T n_l(s,T) \, dW_{s,l}^{\mathbb{Q}}. \tag{A6}$$

Let us now define the random variable

$$\xi_T := -\int_0^T \left(\lambda_t^u + \lambda_t^l\right) dt. \tag{A7}$$

The expected value of this random variable is

$$\mathbb{E}[\xi_T | \mathcal{F}_0] = -\frac{\lambda_0^u}{\kappa_u} n_u(0,T) - \theta_u \int_0^t n_u(s,T) \, ds - \frac{\lambda_0^l}{\kappa_l} n_l(0,T) - \theta_l \int_0^T n_l(s,T) \, ds,$$

and given the variance of the two separate processes—expressed as $\frac{\sigma_u^2}{\kappa_u^2} \int_0^T n_u^2(s,T) \, ds$ and $\frac{\sigma_l^2}{\kappa_l^2} \int_0^T n_l^2(s,T) \, ds$, respectively—then the variance of their sum, that is, the variance of the random variable ξ_T, is given by

$$\frac{\sigma_u^2}{\kappa_u^2} \int_0^t n_u^2(s,T) \, ds + \frac{\sigma_l^2}{\kappa_l^2} \int_0^t n_l^2(s,T) \, ds + 2\rho_{ul} \frac{\sigma_u}{\kappa_u} \frac{\sigma_l}{\kappa_l} \int_0^T n_u(s,T) n_l(s,T) \, ds.$$

Using Musiela and Rutkowski (2005, Lemma 10.1.1), it follows that

$$\begin{aligned}\mathbb{E}[e^{\xi_T} | \mathcal{F}_0] = \exp\Bigg[&-\frac{\lambda_0^u}{\kappa_u} n_u(0,T) - \theta_u \int_0^T n_u(s,T) \, ds \\ &- \frac{\lambda_0^l}{\kappa_l} n_l(0,T) - \theta_l \int_0^T n_l(s,T) \, ds \\ &+ \frac{\sigma_u^2}{2\kappa_u^2} \int_0^T n_u^2(s,T) \, ds + \frac{\sigma_l^2}{2\kappa_l^2} \int_0^T n_l^2(s,T) \, ds \\ &+ \rho_{ul} \frac{\sigma_u}{\kappa_u} \frac{\sigma_l}{\kappa_l} \int_0^T n_u(s,T) n_l(s,T) \, ds \Bigg]. \end{aligned} \tag{A8}$$

Let us now define a new measure, $\tilde{Q}$, such that

$$\frac{d\tilde{Q}}{dQ} = \frac{\mathrm{e}^{-\int_0^T \left(\lambda_t^u + \lambda_t^l\right) dt}}{\mathbb{E}\left[\mathrm{e}^{-\int_0^T \left(\lambda_t^u + \lambda_t^l\right) dt} \middle| \mathcal{F}_0\right]} = \frac{\mathrm{e}^{\xi_T}}{\mathbb{E}[\mathrm{e}^{\xi_T} | \mathcal{F}_0]}, \tag{A9}$$

with

$$
\begin{aligned}
e^{\xi_T} = \exp\Bigg[& -\frac{\lambda_0^u}{\kappa_u} n_u(0,T) - \theta_u \int_0^T n_u(s,T)\,ds \\
& - \frac{\sigma_u}{\kappa_u}\int_0^T n_u(s,T)\,dW_{s,u}^{\mathbb{Q}} \\
& - \frac{\lambda_0^l}{\kappa_l} n_l(0,T) - \theta_l \int_0^T n_l(s,T)\,ds \\
& - \frac{\sigma_l}{\kappa_l}\int_0^T n_l(s,T)\,dW_{s,l}^{\mathbb{Q}}\Bigg],
\end{aligned} \tag{A10}
$$

after combining definition (A7) with Equations (A5) and (A6).

It is possible to observe that there are some common terms between Equations (A8) and (A10). Hence, dividing both the numerator and denominator of Equation (A9) by

$$
\exp\left[-\frac{\lambda_0^u}{\kappa_u} n_u(0,T) - \theta_u \int_0^T n_u(s,T)\,ds - \frac{\lambda_0^l}{\kappa_l} n_l(0,T) - \theta_l \int_0^T n_l(s,T)\,ds\right]
$$

and rearranging yields

$$
\begin{aligned}
\frac{d\tilde{Q}}{dQ} &= \frac{\exp\left[-\frac{\sigma_u}{\kappa_u}\int_0^T n_u(s,T)\,dW_{s,u}^{\mathbb{Q}} - \frac{\sigma_l}{\kappa_l}\int_0^T n_l(s,T)\,dW_{s,l}^{\mathbb{Q}}\right]}{\exp\left[\frac{\sigma_u^2}{2\kappa_u^2}\int_0^T n_u^2(s,T)\,ds + \frac{\sigma_l^2}{2\kappa_l^2}\int_0^T n_l^2(s,T)\,ds + \rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_0^T n_u(s,T) n_l(s,T)\,ds\right]} \\
&= \exp\Bigg[-\frac{\sigma_u}{\kappa_u}\int_0^T n_u(s,T)\,dW_{s,u}^{\mathbb{Q}} - \frac{\sigma_l}{\kappa_l}\int_0^T n_l(s,T)\,dW_{s,l}^{\mathbb{Q}} \\
&\quad -\frac{\sigma_u^2}{2\kappa_u^2}\int_0^T n_u^2(s,T)\,du - \frac{\sigma_l^2}{2\kappa_l^2}\int_0^T n_l^2(s,T)\,du \\
&\quad - \rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_0^T n_u(s,T) n_l(s,T)\,ds\Bigg].
\end{aligned}
$$

The multidimensional Girsanov theorem implies that

$$
W_{t,V}^{\tilde{Q}}(t) = W_{t,V}^{Q} + \frac{\sigma_u \rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds + \frac{\sigma_l \rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds.
$$

Thus, in a new measure $\tilde{Q}$, the value of V_t will be given by

$$
\begin{aligned}
V_t = V_0 \exp\Bigg[& \left(r - q - \frac{1}{2}\sigma_V^2\right)t - \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds \\
& - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds + \sigma_V W_{t,V}^{\tilde{Q}}\Bigg].
\end{aligned} \tag{A11}
$$

Let us now consider the risk-neutral expectation

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[\phi_{\text{dwl}} V_T \mathbb{1}_{\{V_T \le X\}} e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt} \,\middle|\, \mathcal{F}_0\right] \\
&= \phi_{\text{dwl}} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt} \,\middle|\, \mathcal{F}_0\right] \\
&\quad \mathbb{E}_{\mathbb{Q}}\left[V_T \mathbb{1}_{\{V_T \le X\}} \frac{e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt}}{\mathbb{E}\left[e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt} \,\middle|\, \mathcal{F}_0\right]} \,\middle|\, \mathcal{F}_0\right].
\end{aligned} \tag{A12}
$$

The solution to the expected value $\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt} \middle| \mathcal{F}_0\right]$ is given in Equation (A8), and the second expected value under the equivalent measure $\tilde{Q}$ is equal to

$$
\mathbb{E}_{\mathbb{Q}}\left[V_T \mathbb{1}_{\{V_T \le X\}} \frac{e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt}}{\mathbb{E}\left[e^{-\int_0^T (\lambda_t^u + \lambda_t^l)\,dt} \middle| \mathcal{F}_0\right]} \,\middle|\, \mathcal{F}_0\right] = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[V_T \mathbb{1}_{\{V_T \le X\}} | \mathcal{F}_0\right]. \tag{A13}
$$

Then, we define the new equivalent measure $\bar{\mathbb{Q}}$, such that

$$
\frac{d\bar{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} = e^{-\frac{1}{2}\sigma_V^2 T + \sigma_V W_{T,V}^{\tilde{\mathbb{Q}}}}.
$$

With V_t being given as in Equation (A11), Equation (A13) can be written as

$$
\begin{aligned}
& \mathbb{E}_{\tilde{\mathbb{Q}}}\Bigg[V_0 \exp\Bigg[\left(r - q - \frac{1}{2}\sigma_V^2\right)T \\
& - \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\,ds - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\,ds \\
& + \sigma_V W_{T,V}^{\mathbb{Q}}\Bigg] \mathbb{1}_{\{V_T \le X\}} \Bigg| \mathcal{F}_0\Bigg] \\
& = V_0 \exp[(r-q)T] \exp\left[-\frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\,ds\right. \\
& \left. - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\,ds\right] \times \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\exp\left[-\frac{1}{2}\sigma_V^2 T + \sigma_V W_{T,V}^{\mathbb{Q}}\right] \mathbb{1}_{\{V_T \le X\}} | \mathcal{F}_0\right].
\end{aligned}
$$

Changing to the measure $\bar{\mathbb{Q}}$, the above expression is written as

$$
\begin{aligned}
& V_0 \exp[(r-q)T] \exp\left[-\frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\,ds\right. \\
& \left. - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\,ds\right] \times \mathbb{E}_{\bar{\mathbb{Q}}}\left[\mathbb{1}_{\{V_T \le X\}} | \mathcal{F}_0\right] \\
& = V_0 \exp[(r-q)T] \exp\left[-\frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\,ds\right. \\
& \left. - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\,ds\right] \times \left(1 - \mathbb{E}_{\bar{\mathbb{Q}}}\left[\mathbb{1}_{\{V_T > X\}} | \mathcal{F}_0\right]\right),
\end{aligned} \tag{A14}
$$

and by the Girsanov theorem $W_{t,V}^{\bar{\mathbb{Q}}} = W_{t,V}^{\tilde{\mathbb{Q}}} - \sigma_V t$, and under measure $\bar{\mathbb{Q}}$, V_t is

$$
\begin{aligned}
V_t = V_0 \exp\Bigg[& \left(r - q + \frac{1}{2}\sigma_V^2\right)t - \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds \\
& - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds + \sigma_V W_{t,V}^{\bar{\mathbb{Q}}}\Bigg].
\end{aligned}
$$

Therefore, $\log(V_T)$ is normally distributed with expected value

$$
\left(r - q + \frac{1}{2}\sigma_V^2\right)T - \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds
$$

and variance $\sigma_V^2 T$. Thus, we get

$$\mathbb{E}_{\bar{\mathbb{Q}}}\left[\mathbb{1}_{\{V_T>X\}}|\mathcal{F}_0\right] = N\left(\frac{a_1(X) - \frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds - \frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds}{\sigma_V\sqrt{T}}\right), \tag{A15}$$

where N represents the standard normal distribution.

Moreover, combining Equations (A12)–(A15) yields

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\phi_{\mathrm{dwl}}V_T\mathbb{1}_{\{V_T\le X\}}\mathrm{e}^{-\int_0^T\left(\lambda_t^u+\lambda_t^l\right)dt}\middle|\mathcal{F}_0\right]\\
&= \phi_{\mathrm{dwl}}V_0\exp[(r-q)T]\\
&\times\exp\left[-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\,ds - \frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\,ds\right]\\
&\times\exp\left[-\frac{\lambda_0^u}{\kappa_u}n_u(0,T) - \theta_u\int_0^T n_u(s,T)\,ds - \frac{\lambda_0^l}{\kappa_l}n_l(0,T) - \theta_l\int_0^T n_l(s,T)\,ds\right.\\
&+\frac{\sigma_u^2}{2\kappa_u^2}\int_0^T n_u^2(s,T)\,du + \frac{\sigma_l^2}{2\kappa_l^2}\int_0^T n_l^2(s,T)\,du\\
&\left.+\rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_0^T n_u(s,T)n_l(s,T)\,ds\right]\\
&\times\left(1 - N\left(\frac{a_1(X) - \frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_0^T n_u(s,T)\,ds - \frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_0^T n_l(s,T)\,ds}{\sigma_V\sqrt{T}}\right)\right).
\end{aligned}$$

Finally, introducing $\alpha_u \in \{0,1\}$, $\alpha_l \in \{0,1\}$, and $X = D\mathrm{e}^U\mathrm{e}^L$ to accommodate the inclusion of the debt change parameters U and L, the results of Proposition 1 follow immediately. □

Appendix B

Proof of Proposition 2

This proof is similar to that of Proposition 1. The key difference is that now it is not necessary to use the measure $\bar{Q}$. The steps until Equation (A13) are identical. Thus, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[D\mathbb{1}_{\{V_T>X\}}\mathrm{e}^{-\int_0^T\left(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l\right)dt}\middle|\mathcal{F}_0\right]\\
&= D\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T\left(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l\right)dt}\middle|\mathcal{F}_0\right]\mathbb{E}_{\bar{\mathbb{Q}}}\left[\mathbb{1}_{\{V_T>X\}}|\mathcal{F}_0\right],
\end{aligned}$$

where the solution of $\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T\left(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l\right)dt}\middle|\mathcal{F}_0\right]$ is given in Equation (A8), and taking into consideration α_u and α_l we obtain $F(\alpha_u, \alpha_l)$.

As for the remaining expected value, given now $\log(V_T)$ having the expected value of

$$\left(r - q - \frac{1}{2}\sigma_V^2\right)T - \frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds - \frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds$$

and the variance, $\sigma_V T$, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{V_T>X\}}|\mathcal{F}_0\right]\\
&= N\left(\frac{a_2(X) - \alpha_u\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_0^t n_u(s,T)\,ds - \alpha_l\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_0^t n_l(s,T)\,ds}{\sigma_V\sqrt{T}}\right)\\
&:= G(\alpha_u, \alpha_l, a_2(X)).
\end{aligned}$$

□